Mathematics 3333-002 Name (please print) ____________________

Examination II Form A

March 24, 2010

Instructions: Give concise answers, but clearly indicate your reasoning. Most of the problems have rather short answers, so if you find yourself involved in a lengthy calculation, it might be a good idea to move on and come back to that problem if you have time.

I. (8) Consider the vector space $V = \{(x, y, z) \mid x, y, z \text{ are in } \mathbb{R}\}$ with the operations $(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$ and $\lambda \odot (x, y, z) = (x, 1, z)$.

(a) Verify that V satisfies the vector space axiom $\lambda \odot (\mu \odot v) = (\lambda\mu) \odot v$. [Hint: write v as (x, y, z).]

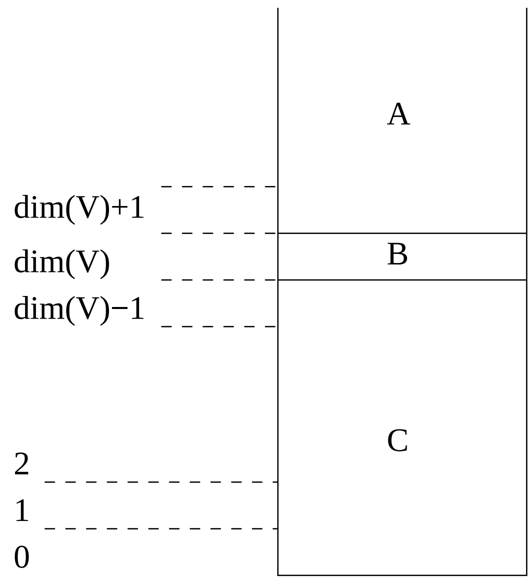
(b) Tell one of the eight vector space axioms that V fails to satisfy, and verify that V fails to satisfy it.

II. (5) The regions A, B, and C in this diagram represent *some* of the finite subsets of the finite subsets of a vector space V, specifically *those which either span, or are linearly independent, or both.* The number of elements in the subsets is indicated by the numbers to the left, 0, 1, 2, ..., $\dim(V) - 1$, $\dim(V)$, $\dim(V) + 1, \ldots$ and so on. For each of the following, tell which region or regions comprise the subsets that:

(a) are linearly independent

(b) span

(c) are bases



III. (5) Using the definition of linear independence, verify that the set $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ is linearly independent.

IV. (4) Define what it means to say that a subset $\{v_1, v_2, \ldots, v_n\}$ of a vector space V is a *basis.* Define the *dimension* of V.

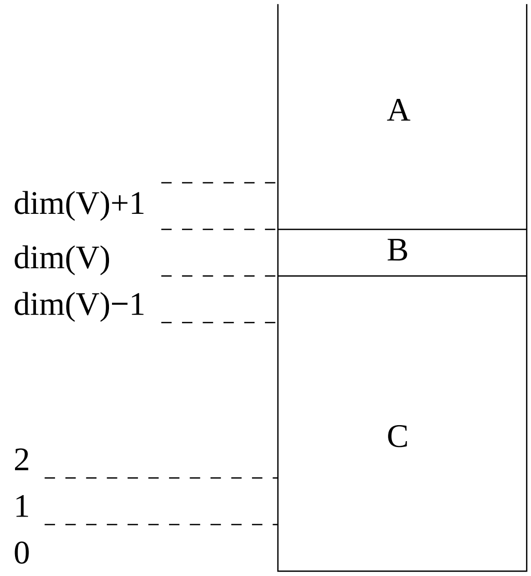
V. (6) The regions A, B, and C in this diagram represents *all* of the finite subsets of a vector space V. The number of elements in the subsets is indicated by the numbers to the left, 0, 1, 2, . . . , $\dim(V)-1$, $\dim(V)$, $\dim(V)+1, \ldots$ and so on. For each of the following, tell which region or regions comprise the subsets that:

(a) might span

(b) cannot span

(c) might be linearly independent

(d) cannot be linearly independent



VI. (5) Recall that if $\{v_1, \ldots, v_k\}$ is a subset of a vector space V, then the span of $\{v_1, \ldots, v_k\}$ is $\text{span}(\{v_1, \ldots, v_k\}) = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \text{ the } \lambda_i \text{ are numbers.}\}$. Verify that $\text{span}(\{v_1, \ldots, v_k\})$ is closed under addition and scalar multiplication.

VII. (8) Find a basis for and the dimension of the solution space of this homogeneous system:

$$\begin{bmatrix} 1 & 3 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

VIII. (6) Find a basis for the row space of the matrix $\begin{bmatrix} 3 & 0 & 1 \\ 3 & -3 & 3 \\ -2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

IX. (8) (a) The rank of a certain 5×4 matrix A is 2. What is the dimension of the solution space of the homogeneous linear system $AX = 0$? Why?
(b) A certain matrix B is the coefficient matrix of a homogeneous linear system of five equations. If the dimension of the solution space of the homogeneous linear system $BX = 0$ is 5, and the dimension of the column space of B is 3, how many variables does the linear system have? Why?