

Instructions: Give concise answers, but clearly indicate your reasoning.

- I.** Let V be an inner product space, that is, a vector space V equipped with an inner product denoted by (u, v) . Let v_0 be a fixed vector, and let $W = \{v \in V \mid (v, v_0) = 0\}$ be the set of vectors in V that are orthogonal to v_0 . Verify that W is a subspace of V .

- II.** Let θ be a fixed real number, and in \mathbb{R}^2 let $v_\theta = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ and $w_\theta = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$. Assuming that \mathbb{R}^2 has the standard inner product, verify that $\{v_\theta, w_\theta\}$ is an orthonormal basis for \mathbb{R}^2 .

- III.** Let $f: V \rightarrow W$ be a linear transformation between two vector spaces.

(6)
(a) Define the *kernel* of f .

(b) Verify that the kernel of f is a subspace of V .

- IV.** Each of the following matrices is the augmented matrix of a system of linear equations, and is in row echelon form or reduced row echelon form. For each matrix, use back substitution to write a general expression for the solutions of the corresponding linear system.

1.
$$\begin{bmatrix} 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & a & b & 1 \\ 0 & 1 & c & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
 (the answer will involve a , b , and c)

V. Let $A = \begin{bmatrix} 0 & 0 & -9 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

- (6)
(a) Calculate the characteristic polynomial of A .
(b) Use the characteristic polynomial to find the only (real) eigenvalue of A .
(c) Find an eigenvector for the eigenvalue.

- VI.** Let $A = ([a_{i,j}])$ be a 5×5 matrix, and consider the formula

(4)
$$\det(A) = \sum (\pm) a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)} a_{4,\sigma(4)} a_{5,\sigma(5)} .$$

Determine the sign (i. e. tell whether the term has a plus or a minus sign in the formula) of the term that contains $a_{1,2} a_{2,4} a_{3,5} a_{4,3} a_{5,1}$ (make your reasoning clear— answers of “plus” or “minus” without a correct explanation won’t receive any credit).

VII. Label each of the following statements either T for true or F for false. The symbol V that appears in some (15) of the statements denotes a finite-dimensional vector space, and $\{v_1, \dots, v_k\}$ denotes a finite subset of V .

_____ If a 5×5 matrix has 5 distinct eigenvalues, then it must be diagonalizable.

_____ If $\{v_1, \dots, v_k\}$ spans V , then some subset of $\{v_1, \dots, v_k\}$ is a basis for V .

_____ If $\{v_1, \dots, v_k\}$ is a basis for V , and two linear transformation f and g from V to W satisfy $f(v_i) = g(v_i)$ for $1 \leq i \leq k$, then $f(v) = g(v)$ for every v in V .

_____ When a homogeneous linear system is written in matrix form $AX = 0$, the null space of A equals the solution space of the linear system.

_____ If $(_, _)$ is an inner product on \mathbb{R}^n , and $c_{ij} = (e_i, e_j)$, then the matrix $C = [c_{ij}]$ satisfies $(v, w) = v^T C w$ for any two vectors v and w in \mathbb{R}^n .

_____ The range of a matrix transformation equals the row space of the matrix.

_____ When a homogeneous linear system is written in matrix form $AX = 0$, the rank of A equals the dimension of the solution space of the linear system.

_____ Similar matrices must have the same characteristic polynomial.

_____ The only matrix that is similar to the identity matrix is the identity matrix.

_____ If a matrix is in row echelon form, then its nonzero rows are linearly independent.

_____ If a matrix is in row echelon form, then its nonzero columns are linearly independent.

_____ A linear transformation is diagonalizable exactly when there is a basis consisting entirely of eigenvectors.

_____ Coordinate vectors satisfy the formulas $(v + w)_S = v_S + w_S$ and $(\lambda v)_S = \lambda v_S$.

_____ When a matrix $[a_{i,j}]$ is lower triangular, its characteristic polynomial is $(\lambda - a_{1,1})(\lambda - a_{2,2}) \cdots (\lambda - a_{n,n})$.

_____ A matrix that has no eigenvectors must be singular.

VIII. Let V be the 2-dimensional vector space consisting of solutions to the differential equation $y'' = y$. Recall (10) that e^x , e^{-x} , $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh(x) = (e^x - e^{-x})/2$ are well-known solutions of this equation.

Let $S = \{e^x, e^{-x}\}$ and $T = \{\cosh(x), \sinh(x)\}$. These are bases of V (you do not need to verify this).

(a) Find $(2e^x - 3e^{-x})_S$ and $(2e^x - 3e^{-x})_T$.

(b) Find the transition matrix $P_{S \leftarrow T}$, and verify that $P_{S \leftarrow T}(2e^x - 3e^{-x})_T = (2e^x - 3e^{-x})_S$.

(c) Let $D: V \rightarrow V$ be differentiation, $D(y) = y'$, which is a linear transformation. Find the representation matrix of D with respect to the basis S , and with respect to the basis T .

IX. A certain 3×3 matrix A has eigenvalues 2, 3, and -1 .

(9) A 2-eigenvector of A is $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, a 3-eigenvector of A is $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and a (-1) -eigenvector of A is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) Calculate $A \left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right)$.

(b) Calculate $A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) Tell a 3×3 matrix P such that $P^{-1}AP$ is a diagonal matrix, and tell the diagonal matrix.

X. Recall that if $\{v_1, \dots, v_k\}$ is a subset of a vector space V , $\text{span}(\{v_1, \dots, v_k\})$ is the set of linear combinations $\{\lambda_1 v_1 + \dots + \lambda_k v_k \mid \text{the } \lambda_i \text{ are numbers.}\}$. Verify that $\text{span}(\{v_1, \dots, v_k\})$ is a subspace of V .

XI. (a) Tell two elementary 3×3 matrices E and F such that $EFA = B$, where $A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 1 & -5 \\ 7 & 0 & 2 \end{bmatrix}$ and

$$B = \begin{bmatrix} 2 & -2 & 6 \\ 4 & 1 & -5 \\ 4 & 3 & -7 \end{bmatrix}$$

(b) Is the matrix $\begin{bmatrix} 2 & -1 & 5 \\ 2 & 1 & -3 \\ 2 & 0 & 1 \end{bmatrix}$ a product of elementary matrices? Why or why not?

XII. (a) Let $\{v_1, \dots, v_k\}$ be a set of vectors in a vector space V . Define what it means to say that $\{v_1, \dots, v_k\}$ is linearly independent.

(b) Test $\left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} \right\}$ for linear independence.