Mathematics 6823-001 Name (please print) ______________________

Midterm Examination

March 12, 2009

Instructions: Insofar as possible, give brief, clear answers. Use major theorems when possible.

**I.** (6) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of abelian groups, and let $G$ be an abelian group. Give an example showing that the sequence $0 \to \mathrm{Hom}(C,G) \xrightarrow{g^*} \mathrm{Hom}(B,G) \xrightarrow{f^*} \mathrm{Hom}(A,G) \to 0$ need not be exact. What positive statement can be made?

**II.** (6) Let $X$ be obtained from the 2-sphere by identifying three points of the equator. Compute the homology groups of $X$. (Note that $X$ has a cell structure with one 0-cell, three 1-cells, and two 2-cells.)

**III.** (6) Let $X$ be a finite CW-complex, and let $A$ and $B$ be subcomplexes of $X$ with $X = A \cup B$. Explain why the Euler characteristic satisfies $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

**IV.** (6) Let $C$ be a chain complex and let $[\varphi] \in H^n(C;G)$.

(a) Use the fact that $\varphi$ is a cocycle to show that $\varphi$ induces a homomorphism $\overline{\varphi|_{Z_n}} \colon H_n(C) \to G$.

(b) Show that if $\varphi$ is a coboundary, then $\overline{\varphi}$ is the zero homomorphism. That is, sending the cohomology class $[\varphi]$ to $\overline{\varphi}$ is a well-defined homomorphism $h \colon H^n(C;G) \to \mathrm{Hom}(H_n(C),G)$.

**V.** (6) Let $H$ be an abelian group (or more generally an $R$-module over a ring $R$). Define a *free resolution* of $H$. Suppose that $F$ and $F'$ are free resolutions of $H$ and $H'$, and $\alpha \colon H \to H'$ is a homomorphism. Tell what is obtained from $\alpha$, and how well-defined it is.

**VI.** (8) State the Excision Theorem (either of the two forms is sufficient). Use it to calculate $H_n(U, U - x)$, where $U$ is an open subset of $\mathbb{R}^n$ and $x \in U$.

**VII.** (4) Construct a surjective map of degree 0 from $S^n$ to $S^n$.

**VIII.** (8) Define the terms *category, covariant functor, and contravariant functor.* Give an elementary (undergraduate) example of a contravariant functor.