I. Let $C$ be a chain complex and let $[\varphi] \in H^n(C;G)$.

(a) Use the fact that $\varphi$ is a cocycle to show that $\varphi$ induces a homomorphism $\overline{\varphi}_{|Z_n}: H_n(C) \to G$.

(b) Show that if $\varphi$ is a coboundary, then $\overline{\varphi}$ is the zero homomorphism. That is, sending the cohomology class $[\varphi]$ to $\overline{\varphi}$ is a well-defined homomorphism $h: H^n(C;G) \to \text{Hom}(H_n(C),G)$.

II. Let $Y$ be the space obtained from the 3-sphere $S^3$ by attaching a 4-cell using a map of degree 6. It has a CW-complex structure with one cell in each of the dimensions 0, 3, and 4.

(a) Use cellular homology to calculate the homology of $Y$ with $\mathbb{Z}$ coefficients.

(b) Use the Universal Coefficient Theorem to calculate the cohomology of $Y$ with $\mathbb{Z}$ coefficients. You may use the fact that $\text{Ext}(\mathbb{Z}/m,G) \cong G/mG$, so $\text{Ext}(\mathbb{Z}/m,\mathbb{Z}) \cong \mathbb{Z}/m$.

(c) Use the Universal Coefficient Theorem to calculate the cohomology of $Y$ with $\mathbb{Z}/3$ coefficients. You may use the fact that $\text{Ext}(\mathbb{Z}/m,G) \cong G/mG$, so $\text{Ext}(\mathbb{Z}/m,\mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m,n)$.

III. Give an example of a short exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ of abelian groups that does not split.

(a) Use an example of a short exact sequence of nonabelian groups so that $\beta$ has a section, but $B$ is not the direct product of $A$ and $C$.

IV. Regarding the Klein bottle $K$ as two Möbius bands glued together along their boundaries, use the Mayer-Vietoris sequence to calculate the homology groups of $K$.

V. Give the definitions of $\varphi \cup \psi$, where $\varphi \in C^k(X;G)$ and $\psi \in C^\ell(X;G)$, and of $\sigma \cap \varphi$, where $\sigma$ is a singular $(k+\ell)$-simplex in $X$.

VI. Use Poincaré Duality to show that if $M$ is a closed odd-dimensional manifold, then the Euler characteristic of $M$ is 0. You may use the fact that $H^*(M;F) \cong \text{Hom}(H_*(M;F),F)$ when $F$ is a field, and also the fact that the $H_i(M;F)$ are finite-dimensional.

VII. Recall that if $X$ and $Y$ are CW-complexes and each $H^k(Y;R)$ is free and finitely generated (as an $R$-module), then $H^*(X \times Y;R) \cong H^*(X;R) \otimes H^*(Y;R)$ as graded rings. Take as known the fact that for $n \geq 1$, $H^*(S^n) \cong H^0(S^n) \oplus H^n(S^n)$, with $H^0(S^n) \cong \mathbb{Z}$ generated by 1, $H^n(S^n) \cong \mathbb{Z}$ generated by an element $\alpha_n$, and $\alpha_n \cup \alpha_n = 0$.

(a) Use this theorem to write down the cohomology ring $H^*(S^n \times S^n)$.

(b) Tell a ring isomorphism (not a graded ring isomorphism) from $H^*(S^2 \times S^2)$ to $H^*(S^4 \times S^4)$. You do not need to verify that it is is an isomorphism, just write it down.

(c) Show that $H^*(S^2 \times S^2)$ and $H^*(S^3 \times S^3)$ are not isomorphic as rings.

VIII. Let $U$ be an open subset of an $R$-orientable $n$-manifold $M$, and let $\{\mu_x\}_{x \in M}$ be an $R$-orientation of $M$.

(a) Verify that $\{\mu_x\}_{x \in U}$ is an $R$-orientation of $U$ (i.e. is locally consistent for $U$).
IX. Let $M$ be a closed connected $R$-orientable $n$-manifold. What is a fundamental class for $M$ (with $R$ coefficients)? State Poincaré Duality for this closed $M$ in terms of a fundamental class.

X. Let $f: (I^n, \partial I^n) \to (X, x_0)$ represent an element of $\pi_n(X, x_0)$, and let $\omega: (I, \partial I) \to (X, x_0)$ be a loop based at $x_0$. Draw a picture and use it to describe a function $\omega f: (I^n, \partial I^n) \to (X, x_0)$ that represents the result $[\omega] \cdot [f]$ of the element $[\omega] \in \pi_1(X, x_0)$ acting on the element $[f] \in \pi_n(X, x_0)$. Do the same if one is thinking of $f$ as a map from $(S^n, s_0) \to (X, x_0)$.

XI. Let $M$ be a simply-connected $n$-manifold. Show that $M$ is orientable. (Hint: what do we know about the covering spaces of a simply-connected space?)