Math 6823 homework

1. (due 2/3) 2.1 # 28, 29

2. (2/3) 2.1 # 31

3. (2/3) 2.2 # 4

4. (2/12) 2.2 # 3, 5 (the reflection in a hyperplane H with $0 \in H$ sends x to $x - 2\langle x, u\rangle u$ where u is a unit normal to H), 6

5. (2/12) 2.2 # 12, 13 (to prove $X \simeq S^2$, one can apply Proposition 0.18), 15

6. (2/12) 2.2 # 20 (induct on the number of cells of X), 23 (you may assume the result in problem 22)

7. (2/26) 2.2 # 27, 28, 32

8. (2/26) 2.3 # 1, 2

9. (3/5) First convince youself that for a homomorphism $h\colon \mathbb{Z}^m \to \mathbb{Z}^n$ given by a matrix A, the matrix of $h^*\colon \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z}^m, \mathbb{Z})$ with respect to the dual bases is the transpose of A, and the matrix of $h^*\colon \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}/2) \to \operatorname{Hom}(\mathbb{Z}^m, \mathbb{Z}/2)$ is the mod 2 reduction of the transpose. Then do 3.1 # 6 for the Klein bottle case (do the torus and/or projective plane case if you have time and think it is worthwhile).

10. (3/5) 3.1 # 1 (For the contravariant functor, use the Fundamental Lemma. For the covariant one, given a homomorphism $\beta\colon G \to G'$, define $\beta_\#\colon \operatorname{Hom}(F_i, G) \to \operatorname{Hom}(F_i, G')$ by $\beta_\#(\phi) = \beta\circ\phi$. Check that this is a chain map, so induces $\beta_*\colon H^n(F; G) \to H^n(F; G')$ defined by $\beta_*[\phi] = [\beta_\#(\phi)]$. You will need to check the functorial properties, which is straightforward), 2 (use problem 1), 3 (This problem gives a relatively simple example of nontrivial higher Ext groups, where $\operatorname{Ext}^n_R(H, G)$ is defined on page 197. In this problem, $\mathbb{Z}/2$ is regarded as a $\mathbb{Z}/4$-module just by $a \cdot b = ab \in \mathbb{Z}/2$ for $a \in \mathbb{Z}/4 = \{0, 1, 2, 3\}$ and $b \in \mathbb{Z}/2 = \{0, 1\}$. There is a $\mathbb{Z}/4$-module homomorphism $p\colon \mathbb{Z}/4 \to \mathbb{Z}/2$ defined by reduction mod 2. A free $\mathbb{Z}/4$ resolution of $\mathbb{Z}/2$ is obtained by taking each $F_i \cong \mathbb{Z}/4$ and each $f_i = q$, where $q(x) = 2x$. Now you just need to work out $\operatorname{Hom}(\mathbb{Z}/4, \mathbb{Z}/2)$ and q^* to compute $\operatorname{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$.)

11. (4/2) 3.1 # 9 (use the Universal Coefficient Theorem)

12. (4/2) 3.2 # 2

13. (4/2) 3.2 # 3 (Just do the case $n \geq 3$. The map $g\colon S^n \to S^{n-1}$ induces $\overline{g}\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$. First show that $\overline{g}_{\#}\colon \pi_1(\mathbb{RP}^n) \to \pi_1(\mathbb{RP}^{n-1})$ is an isomorphism; both of the fundamental groups are $\mathbb{Z}/2$, lift a generator α to a path in S^n, observe that g sends it to a lift of the nontrivial element of $\pi_1(\mathbb{RP}^{n-1})$, and argue from there. Next, since H_1 is the abelianization of π_1, we can conclude that $\overline{g}_*\colon H_1(\mathbb{RP}^n) \to H_1(\mathbb{RP}^{n-1})$ is an isomorphism. Now, use the commutative diagram whose vertical arrows are the isomorphisms $h\colon H^1(\mathbb{RP}^n;\mathbb{Z}/2) \to \mathrm{Hom}(H_1(\mathbb{RP}^n),\mathbb{Z}/2)$ and similarly for $\mathbb{RP}^{n-1}$ to deduce that $\overline{g}^*\colon H^1(\mathbb{RP}^{n-1};\mathbb{Z}/2) \to H^1(\mathbb{RP}^n;\mathbb{Z}/2)$ is an isomorphism.)

14. (4/2) 3.2 # 6 (for the case $n = 1$, use the homeomorphism from $\mathbb{CP}^1 = S^2$ to $\mathbb{C} \cup \{\infty\}$ that sends $[z_0, z_1]$ to z_0/z_1)

15. (4/16) 3.2 # 11

16. (4/16) 3.3 # 2 (first observe that if $p\colon \widetilde{M} \to M$ is the orientable double cover, then the orientable double cover of $M - x$ is $\widetilde{M} - p^{-1}(x)$, then use the fact that the orientable double covering is connected if and only if M is nonorientable), 3 (use the local homeomorphism property and excisions to get isomorphisms $H_n(\widetilde{M}|\widetilde{x}) \cong H_n(M|x)$, and use these to lift the μ_x in the orientation of M to local orientations in $\widetilde{M}$; for local consistency use the fact that each open ball B is evenly covered and if $\widetilde{B}$ is a component of the preimage of B then we have a natural isomorphism $H_n(\widetilde{M}|\widetilde{B}) \to H_n(M|B)$ given by excisions and $(p|_{\widetilde{B}})_*$), 4 (this one is completely optional, don't worry about it if you already have spent significant time on problems 2 and 3; use the local homology isomorphisms in problem 3, this time to define local orientations on M/G; the fact that the action is orientation-preserving says that for each covering transformation g and each $x \in M$, $g_*\colon H_n(M|x) \to H_n(M|g(x))$ takes μ_x to $\mu_{g(x)}$, so that the local orientations defined in M/G do not depend on the choice of point in the preimage).

17. (4/30) 3.3 # 7 (In this problem, a fundamental class is a generator of $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. To define the map $f\colon M \to S^n$, choose a closed n-ball D in M, let B be its interior, and use the quotient map $M \to M/(M - B) \approx D/\partial D \approx S^n$. Then make use of Theorem 3.26.), 16 (You may assume that α is a single k-simplex, and just verify the formula at the chain level using the definitions— you need not verify all the module properties. The main point of the problem is that the formula says that $(xr)s = x(rs)$ for $x \in H_*(X)$ and $r, s \in H^*(X)$. The other module properties come from R-bilinearity of the cap product.)

18. (4/30) 3.3 # 20

19. (4/30) Verify that if I_0 is a cofinal subset of a directed set I, and $\{G_\alpha\}_{\alpha\in I}$ is a directed system, then $\varinjlim \{G_\alpha\}_{\alpha\in I_0}$ is isomorphic to $\varinjlim \{G_\alpha\}_{\alpha\in I}$.