Mathematics 3333-001 Name (please print) ____________________

Examination III

April 28, 2009

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (8) Let A be the matrix $\begin{bmatrix} 2 & 4 & -2 & 0 \\ 1 & 3 & -1 & -3 \\ 2 & 5 & -2 & -3 \end{bmatrix}$.

(a) Find a basis for the row space of A.

Using elementary row operations, we put A into row echelon form:

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 1 & 3 & -1 & -3 \\ 2 & 5 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis for the row space is $\{\begin{bmatrix} 1 & 0 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -3 \end{bmatrix}\}$.

(b) Find a basis for the column space of A.

Using elementary column operations, we put A into column echelon form:

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 1 & 3 & -1 & -3 \\ 2 & 5 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \end{bmatrix}$$

So a basis for the column space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

II. (6) A certain 14×10 matrix A has rank 8.

(a) What is the dimension of the solution space of the homogeneous system $AX = 0$?

It is 2. The solution space of $AX = 0$ is the null space of A, so the dimension of the solution space is the nullity of A. Since A is 14×10, its rank plus its nullity equals 10, so the nullity must be 2.

(b) Can one solve the linear system $AX = B$ for all choices of B? Why or why not?

No. The rank is the dimension of the column space, which is the set of B so that $AX = B$ has a solution. Since the rank is only 8, the column space is not all of $\mathbb{R}^{14}$.

III. (7) Let V be a vector space with an inner product $(\_,\_)$.

(a) Let w_0 be a fixed vector in V. Show that the set of vectors orthogonal to w_0 is a subspace of V.

Suppose that v_1 and v_2 are orthogonal to w_0, so $(w_0, v_1) = 0$ and $(w_0, v_2) = 0$. Then $(w_0, v_1 + v_2) = (w_0, v_1) + (w_0, v_2) = 0$, so $v_1 + v_2$ is also orthogonal to w_0. For any scalar λ, $(w_0, \lambda v_1) = \lambda(w_0, v_1) = 0$, so λv_1 is also orthogonal to w_0.

(b) Define what it means to say that a set of vectors $S = \{v_1, \ldots, v_n\}$ in V is *orthogonal.*

It means that each v_i is nonzero, and $(v_i, v_j) = 0$ whenever i is not equal to j.

(c) Define what it means to say that a set of vectors $S = \{v_1, \ldots, v_n\}$ in V is *orthonormal.*

It means that S is orthogonal, and moreover each $\| v_i \| = 1$, that is, each $(v_i, v_i) = 1$.

IV. (3) By counting the number of inversions in the permuation 48253176 of eight elements, determine whether this permutation is even or odd.

The initial 4 appears before 3 smaller numbers, so gives 3 inversions. Similarly 8 gives 6 inversions, 2 gives 1, 5 gives 2, 3 gives 1, 1 gives none, 7 gives 1, and 6 gives none, for a total of $3+6+1+2+1+1 = 14$ inversions. Since this is an even number, the permutation is an even permutation.

V. (3) Use the *row operation* method to calculate the determinant $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 6 & 2 & -1 \end{vmatrix}$.

Using Type III row operations, we have something such as

$$\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 6 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 0 & -6 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 5 & 0 \\ 0 & -6 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 7 & -4 \\ 0 & -6 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & -9 \\ 0 & -6 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & -59 \end{vmatrix} = -59$$

(If you use Type I or Type II operations, suitable signs or scalar factors must be introduced at those steps.)

VI. (6) Let $A = \begin{bmatrix} t & 1 & 2 \\ -t & 1 & 1 \\ 0 & 2 & t \end{bmatrix}$.

(a) Calculate that $\det(A) = 2t(t-3)$ by using *cofactor expansion* of the determinant *down the first column.*

$$\begin{vmatrix} t & 1 & 2 \\ -t & 1 & 1 \\ 0 & 2 & t \end{vmatrix} = t\begin{vmatrix} 1 & 1 \\ 2 & t \end{vmatrix} - (-t)\begin{vmatrix} 1 & 2 \\ 2 & t \end{vmatrix} = t(t-2) + t(t-4) = t(t-2+t-4) = 2t(t-3)$$

(b) Use the expression for $\det(A)$ in part (a) (even if you did not carry out the calculation) to determine the values of t for which A is singular.

A is singular exactly when $\det(A) = 0$, that is, when $t = 0$ or $t = 3$.

VII. (4) As usual, let P_3 be the 4-dimensional vector space of polynomials of degree at most 3. Let $D\colon P_3 \to P_3$ be differentiation, a linear transformation. What is the kernel of D? What is the range of D? (Remark: there is no need to use a matrix representation. Just think about what polynomials are in the kernel and range.)

The kernel consists of the polynomials with derivative the zero function, that is the constant polynomials, and the range is P_2. (The range is all the polynomials that are the derivative of a polynomial of degree 3 or less. Such a derivative is of degree at most 2, so is contained in P_2. On the other hand, $t^2 = D(t^3/3)$, $t = D(t^2/2)$, and $1 = D(t)$, so the range contains $\text{span}\{t^2, t, 1\} = P_2$. So the range is the subspace P_2 of P_3.)

VIII. (15) Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - z \\ 2x - y \\ x + 2y - z \end{bmatrix}$$

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(a) Find the *standard* matrix representation for L, that is, find the matrix A so that $AX = L(X)$ for every X in $\mathbb{R}^3$.

$$L(e_1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, L(e_2) = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, L(e_3) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \text{ so } A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

(b) Consider the ordered basis $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$. By solving the system

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

for λ_1, λ_2, and λ_3, obtain a formula of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_S = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$ to convert a vector v in $\mathbb{R}^3$ into its corresponding vector v_S.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 - \lambda_3 \\ \lambda_1 + 2\lambda_2 - \lambda_3 \\ 2\lambda_2 + \lambda_3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & -1 & a \\ 1 & 2 & -1 & b \\ 0 & 2 & 1 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 4a - 3b + c \\ 0 & 1 & 0 & -a + b \\ 0 & 0 & 1 & 2a - 2b + c \end{bmatrix} \text{ so } \begin{bmatrix} a \\ b \\ c \end{bmatrix}_S = \begin{bmatrix} 4a - 3b + c \\ -a + b \\ 2a - 2b + c \end{bmatrix}$$

(c) Check that the formula that you found in part (b) really does convert the three vectors in the basis S to the standard basis vectors, that is, it should tell you that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and similarly for the second and third vectors in S. If it does not, go back and do part (b) correctly.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_S = \begin{bmatrix} 4 - 3 \\ -1 + 1 \\ 2 - 2 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_S = \begin{bmatrix} 4 - 6 + 2 \\ -1 + 2 \\ 2 - 4 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}_S = \begin{bmatrix} -4 - 3(-1) + 1 \\ -(-1) + (-1) \\ 2(-1) - 2(-1) + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(d) Find the matrix representation of L *with respect to the basis* S given in part (b). That is, find the matrix A so that $Av_S = (L(v))_S$.

We want A so that $A(s_i)_S = (L(s_i))_S$ for the vectors $s_i \in S$. Since $A(s_i)_S = Ae_i$, the i^{th} column of A will be $(L(s_i)_S$.

$$L(s_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, L(s_2) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, L(s_3) = \begin{bmatrix} -3 \\ -1 \\ -4 \end{bmatrix}$$

$$(L(s_1))_S = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}, (L(s_2))_S = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, (L(e_3))_S = \begin{bmatrix} -13 \\ 2 \\ -8 \end{bmatrix}, \text{ so } A = \begin{bmatrix} 8 & 3 & -13 \\ -1 & 0 & 2 \\ 5 & 3 & -8 \end{bmatrix}$$