I. Use the row operation method to calculate the inverse of the matrix \[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
0 & 1 & 1
\end{pmatrix}.
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 3 & 0 & 1 \\
0 & 1 & 1 & 0 & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 2 & -1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 2 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 & -1 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & 1 & -1 & 2 \\
0 & 0 & 1 & -1 & 1 & -1
\end{pmatrix}
\]

So the inverse is
\[
\begin{pmatrix}
1 & 0 & -1 \\
1 & -1 & 2 \\
-1 & 1 & -1
\end{pmatrix}.
\]

II. Let \( A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

(a) Show that \( A \) and \( B \) are row equivalent. Give a list of elementary matrices \( E_1, \ldots, E_k \) for which \( E_k \cdots E_1 A = B \).

A sequence of row operations taking \( A \) to \( B \) is \( R_2 - 2R_1 \rightarrow R_2, R_2 \leftrightarrow R_3, (-1)R_1 \rightarrow R_1 \). Performing these same column operations to \( I_3 \) gives
\[
E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

so \( E_3 E_2 E_1 A = B \).

(b) Explain why \( A \) and \( B \) cannot be column equivalent.

The matrix \( B \) has all zero entries in its third row. No sequence of column operations on \( B \) will ever create a nonzero entry in the third row—interchanging and multiplying by nonzero numbers cannot change the third row, and adding a multiple of 0 to 0 still gives 0. So the 1 in the third row of \( A \) can never be obtained. (Alternatively, one can explain that no sequence of column operations on \( A \) can ever get rid of all the nonzero entries).
III. Let $V$ be a vector space and let $S = \{v_1, \ldots, v_k\}$ be a subset of $V$. Recall that $\text{span}(S)$ is the set of all linear combinations of elements of $S$, that is, \( \{ \sum_{i=1}^{k} \lambda_i v_i \mid \lambda_i \in \mathbb{R} \} \). Verify that $\text{span}(S)$ is a subspace of $V$.

Suppose that $a_1 v_1 + \cdots + a_k v_k$ and $b_1 v_1 + \cdots + b_k v_k$ are in $\text{span}(S)$. Then
\[
(a_1 v_1 + \cdots + a_k v_k) + (b_1 v_1 + \cdots + b_k v_k) = (a_1 + b_1) v_1 + \cdots + (a_k + b_k) v_k,
\]
which is also in $\text{span}(S)$. Now, suppose that $\lambda \in \mathbb{R}$. Then
\[
\lambda (a_1 v_1 + \cdots + a_k v_k) = (\lambda a_1) v_1 + \cdots + (\lambda a_k) v_k,
\]
which is also in $\text{span}(S)$.

IV. Let $W = \{ a t^2 + b t + c \mid c \geq 0 \}$, that is, the set of all polynomials of degree at most 2 and having non-negative constant term. By giving a specific counterexample, show that $W$ is not a subspace of $P_2$ (the vector space of all polynomials of degree at most 2).

\[
x^2 + 2 \in W, \text{ but } -2(x^2 + 2) = -2x^2 - 4 \notin W \text{ (or any similar example of multiplying an element of } W \text{ with positive constant term by a negative scalar)}
\]

V. Let $A$ be an $m \times n$ matrix and consider the homogeneous system of linear equations given by $AX = 0$. Its solutions form a subset of $\mathbb{R}^n$. Verify that the set of solutions is a subspace of $\mathbb{R}^n$.

Suppose $X_1$ and $X_2$ are solutions, so $AX_1 = 0$ and $AX_2 = 0$. Then $A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$, so $X_1 + X_2$ is a solutions. Suppose $X_1$ is a solution and $\lambda \in \mathbb{R}$. Then $A(\lambda X_1) = \lambda AX_1 = \lambda 0 = 0$, so $\lambda X_1$ is also a solution.

VI. Let $V$ be a vector space and let $S = \{v_1, \ldots, v_k\}$ be a subset of $V$.

(a) Define what it means to say that $S$ is linearly independent.

$S$ is linearly independent if $\sum_{i=1}^{k} \lambda_i v_i = 0$ only when all $\lambda_i = 0$.

(b) Define what it means to say that $S$ is a basis of $V$.

$S$ is a basis if it spans $V$ and is linearly independent.

(c) If $V$ has dimension 6 and $S$ is a subset consisting of five elements of $V$, what can you say about $S$, beyond just the fact that it is not a basis?

We can say that $S$ does not span $V$.

(d) If $V$ has dimension 6 and $S$ is a subset consisting of seven elements of $V$, what can you say about $S$, beyond just the fact that it is not a basis?

We can say that $S$ is not linearly independent.
VII. Let $V = \mathbb{R}^3$ (the vector space of $1 \times 3$ vectors), and let $S = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 5 & 2 & 3 \end{bmatrix} \right\}$. Test $S$ for linear independence. If it is not linearly independent, write one of its elements as a linear combination of the others.

Suppose we have
\[
\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 & 2 & 3 \end{bmatrix} + \lambda_3 \begin{bmatrix} 5 & 2 & 3 \end{bmatrix}
\]
\[
= \begin{bmatrix} \lambda_1 + 2\lambda_2 + 5\lambda_3 & 2\lambda_1 + 2\lambda_2 + 3\lambda_3 & 3\lambda_1 + 3\lambda_2 + 3\lambda_3 \end{bmatrix}
\]
Using Gaussian elimination to solve these linear equations for the $\lambda_i$, we have
\[
\begin{pmatrix}
1 & 2 & 5 & 0 \\
2 & 2 & 2 & 0 \\
3 & 3 & 3 & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & -3 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
so a general solution is $(3r, -4r, r)$. Since there are nonzero solutions, the vectors are not linearly independent.

Taking, say, $r = 1$ gives the solution $(\lambda_1, \lambda_2, \lambda_3) = (3, -4, 1)$ so
\[
\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = 3 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 2 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \end{bmatrix}
\]
Solving for $\begin{bmatrix} 5 & 2 & 3 \end{bmatrix}$ we have
\[
\begin{bmatrix} 5 & 2 & 3 \end{bmatrix} = -3 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + 4 \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}
\]

VIII. If an $n \times n$ nonsingular matrix $A$ is equivalent to a matrix $B$, then $B$ must also be nonsingular. Why?

If $A$ is equivalent to $B$, then $B = PAQ$ for some nonsingular matrices $P$ and $Q$. Since $A$ is also nonsingular, the product $PAQ$ is nonsingular. That is, $B$ is nonsingular.

(Alternatively, one can explain that performing an elementary row or column operation on a nonsingular matrix can never produce a singular matrix, since the result is either $EA$ or $AE$ with $A$ nonsingular and $E$ elementary and hence nonsingular, so the result is a product of nonsingular matrices and therefore is nonsingular.)

IX. If $P$ is a nonsingular $n \times n$ matrix, then its transpose $P^T$ must also be nonsingular. Why?

When $P$ is nonsingular, it has an inverse $P^{-1}$ with $PP^{-1} = I$. Taking transposes, and because $I$ is symmetric, we have $(PP^{-1})^T = I^T = I$, that is, $(P^{-1})^T P^T = I$. So $P^T$ is nonsingular, and its inverse is $(P^{-1})^T$. 

X. Let $V$ be the vector space of all differentiable functions from the real numbers to the real numbers, with the usual addition and scalar multiplication operations.

(a) Verify that the subset $\{1, x, x^2, x^3\}$ is a linearly independent subset of $V$ (hint: suppose you have the zero function 0 written as a linear combination of these functions, then take derivatives three times).

$$
\begin{align*}
\lambda_0 \cdot 1 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 &= 0 \\
\lambda_1 + 2\lambda_2 x + 3\lambda_3 x^2 &= 0 \\
2\lambda_2 + 6\lambda_3 x &= 0 \\
6\lambda_3 &= 0 \\
\lambda_3 &= 0 \\
\lambda_0 \cdot 1 + \lambda_1 x + \lambda_2 x^2 &= 0 \\
\lambda_1 + 2\lambda_2 x &= 0 \\
2\lambda_2 &= 0 \\
\lambda_2 &= 0 \\
\lambda_0 \cdot 1 + \lambda_1 x &= 0 \\
\lambda_1 &= 0 \\
\lambda_0 \cdot 1 &= 0 \\
\lambda_0 &= 0
\end{align*}
$$

Or alternatively

$$
\begin{align*}
\lambda_0 \cdot 1 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 &= 0 \\
\lambda_0 &= 0 \\
\lambda_1 + 2\lambda_2 x + 3\lambda_3 x^2 &= 0 \\
\lambda_1 &= 0 \\
2\lambda_2 + 6\lambda_3 x &= 0 \\
6\lambda_3 &= 0 \\
\lambda_3 &= 0
\end{align*}
$$

Putting $x = 0$ says that $\lambda_0 = 0$. Taking the derivative gives

$$
\begin{align*}
\lambda_1 + 2\lambda_2 x + 3\lambda_3 x^2 &= 0 \\
\lambda_1 &= 0 \\
2\lambda_2 + 6\lambda_3 x &= 0 \\
2\lambda_2 &= 0 \\
\lambda_2 &= 0 \\
\lambda_0 \cdot 1 &= 0 \\
\lambda_0 &= 0
\end{align*}
$$

Putting $x = 0$ says that $\lambda_2 = 0$. Taking the derivative gives

$$
\begin{align*}
6\lambda_3 &= 0 \\
\lambda_3 &= 0
\end{align*}
$$

(b) The same kind of argument as in (a) can be used to show that the set $\{1, x, x^2, x^3, x^4\}$ is a linearly independent subset of $V$, and even that the sets $\{1, x, x^2, x^3, \ldots, x^n\}$ are linearly independent for any choice of $n$ (do not try to check these facts). What does this tell us about the dimension of $V$. Why?

It shows that $V$ does not have finite dimension. For if it did, the dimension would be a maximum size for a linearly independent subset.