Mathematics 3333-001 Name (please print) ______________________

Examination II

March 24, 2009

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (6) Use the row operation method to calculate the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$

So the inverse is $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$.

II. (7) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Show that A and B are row equivalent. Give a list of elementary matrices $E_1, \ldots, E_k$ for which $E_k \cdots E_1 A = B$.

A sequence of row operations taking A to B is $R_2 - 2R_1 \to R_2$, $R_2 \leftrightarrow R_3$, $(-1)R_1 \to R_1$. Performing these same column operations to I_3 gives

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so $E_3 E_2 E_1 A = B$.

(b) Explain why A and B cannot be column equivalent.

The matrix B has all zero entries in its third row. No sequence of column operations on B will ever create a nonzero entry in the third row— interchanging and muultiplying by nonzero numbers cannot change the third row, and adding a multiple of 0 to 0 still gives 0. So the 1 in the third row of A can never be obtained. (Alternatively, one can explain that no sequence of column operations on A can ever get rid of all the nonzero entries).

III. (4) Let V be a vector space and let $S = \{v_1, \ldots, v_k\}$ be a subset of V. Recall that span(S) is the set of all linear combinations of element of S, that is, $\{\sum_{i=1}^{k} \lambda_i v_i \mid \lambda_i \in \mathbb{R}\}$. Verify that span$(S)$ is a subspace of V.

Suppose that $a_1v_1 + \cdots + a_kv_k$ and $b_1v_1 + \cdots + b_kv_k$ are in span(S). Then

$$(a_1v_1 + \cdots + a_kv_k) + (b_1v_1 + \cdots + b_kv_k) = (a_1 + b_1)v_1 + \cdots + (a_k + b_k)v_k \ ,$$

which is also in span(S). Now, suppose that $\lambda \in \mathbb{R}$. Then

$$\lambda(a_1v_1 + \cdots + a_kv_k) = (\lambda a_1)v_1 + \cdots + (\lambda a_k)v_k \ ,$$

which is also in span(S).

IV. (3) Let $W = \{at^2 + bt + c \mid c \geq 0\}$, that is, the set of all polynomials of degree at most 2 and having non-negative constant term. By giving a specific counterexample, show that W is not a subspace of P_2 (the vector space of all polynomials of degree at most 2).

$x^2 + 2 \in W$, but $-2(x^2 + 2) = -2x^2 - 4 \notin W$ (or any similar example of multiplying an element of W with positive constant term by a negative scalar)

V. (4) Let A be an $m \times n$ matrix and consider the homogeneous system of linear equations given by $AX = 0$. Its solutions form a subset of $\mathbb{R}^n$. Verify that the set of solutions is a subspace of $\mathbb{R}^n$.

Suppose X_1 and X_2 are solutions, so $AX_1 = 0$ and $AX_2 = 0$. Then $A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0$, so $X_1 + X_2$ is a solutions. Suppose X_1 is a solution and $\lambda \in \mathbb{R}$. Then $A(\lambda X_1) = \lambda\, AX_1 = \lambda\, 0 = 0$, so λX_1 is also a solution.

VI. (9) Let V be a vector space and let $S = \{v_1, \ldots, v_k\}$ be a subset of V.

(a) Define what it means to say that S is *linearly independent.*

S is linearly independent if $\sum_{i=1}^{k} \lambda_i v_i = 0$ only when all $\lambda_i = 0$.

(b) Define what it means to say that S is a *basis* of V.

S is a basis if it spans V and is linearly independent.

(c) If V has dimension 6 and S is a subset consisting of five elements of V, what can you say about S, beyond just the fact that it is not a basis?

We can say that S does not span V.

(d) If V has dimension 6 and S is a subset consisting of seven elements of V, what can you say about S, beyond just the fact that it is not a basis?

We can say that S is not linearly independent.

VII. (6) Let $V = \mathbb{R}_3$ (the vector space of 1×3 vectors), and let $S = \left\{ \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \right\}$. Test S for linear independence. If it is not linearly independent, write one of its elements as a linear combination of the others.

Suppose we have

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 & 2 & 3 \end{bmatrix} + \lambda_3 \begin{bmatrix} 5 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 + 2\lambda_2 + 5\lambda_3 & 2\lambda_1 + 2\lambda_2 + 2\lambda_3 & 3\lambda_1 + 3\lambda_2 + 3\lambda_3 \end{bmatrix}$$

Using Gaussian elimination to solve these linear equations for the λ_i, we have

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so a general solution is $(3r, -4r, r)$. Since there are nonzero solutions, the vectors are not linearly independent.

Taking, say, $r = 1$ gives the solution $(\lambda_1, \lambda_2, \lambda_3) = (3, -4, 1)$ so

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = 3 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 2 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \end{bmatrix}$$

Solving for $\begin{bmatrix} 5 & 2 & 3 \end{bmatrix}$ we have

$$\begin{bmatrix} 5 & 2 & 3 \end{bmatrix} = -3 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + 4 \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$$

VIII. (4) If an $n \times n$ nonsingular matrix A is equivalent to a matrix B, then B must also be nonsingular. Why?

If A is equivalent to B, then $B = PAQ$ for some nonsingular matrices P and Q. Since A is also nonsingular, the product PAQ is nonsingular. That is, B is nonsingular.

(Alternatively, one can explain that performing an elementary row or column operation on a nonsingular matrix can never produce a singular matrix, since the result is either EA or AE with A nonsingular and E elementary and hence nonsingular, so the result is a product of nonsingular matrices and therefore is nonsingular.)

IX. (4) If P is a nonsingular $n \times n$ matrix, then its transpose P^T must also be nonsingular. Why?

When P is nonsingular, it has an inverse P^{-1} with $PP^{-1} = I$. Taking transposes, and because I is symmetric, we have $(PP^{-1})^T = I^T = I$, that is, $(P^{-1})^T P^T = I$. So P^T is nonsingular, and its inverse is $(P^{-1})^T$.

X. (6) Let V be the vector space of all differentiable functions from the real numbers to the real numbers, with the usual addition and scalar multiplication operations.

(a) Verify that the subset $\{1, x, x^2, x^3\}$ is a linearly independent subset of V (hint: suppose you have the zero function 0 written as a linear combination of these functions, then take derivatives three times).

$$\lambda_0 \cdot 1 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0$$
$$\lambda_1 + 2\lambda_2 x + 3\lambda_3 x^2 = 0$$
$$2\lambda_2 + 6\lambda_3 x = 0$$
$$6\lambda_3 = 0$$
$$\lambda_3 = 0$$

$$\lambda_0 \cdot 1 + \lambda_1 x + \lambda_2 x^2 = 0$$
$$\lambda_1 + 2\lambda_2 x = 0$$
$$2\lambda_2 = 0$$
$$\lambda_2 = 0$$

$$\lambda_0 \cdot 1 + \lambda_1 x = 0$$
$$\lambda_1 = 0$$

$$\lambda_0 \cdot 1 = 0$$
$$\lambda_0 = 0$$

Or alternatively

$$\lambda_0 \cdot 1 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0$$
Putting $x = 0$ says that $\lambda_0 = 0$. Taking the derivative gives
$$\lambda_1 + 2\lambda_2 x + 3\lambda_3 x^2 = 0$$
Putting $x = 0$ says that $\lambda_1 = 0$. Taking the derivative gives
$$2\lambda_2 + 6\lambda_3 x = 0$$
Putting $x = 0$ says that $\lambda_2 = 0$. Taking the derivative gives
$$6\lambda_3 = 0 \text{ so } \lambda_3 = 0.$$

(b) The same kind of argument as in (a) can be used to show that the set $\{1, x, x^2, x^3, x^4\}$ is a linearly independent subset of V, and even that the sets $\{1, x, x^2, x^3, \ldots, x^n\}$ are linearly independent for any choice of n (do not try to check these facts). What does this tell us about the dimension of V. Why?

It shows that V does not have finite dimension. For if it did, the dimension would be a maximum size for a linearly independent subset.