I. Use the row operation method to calculate the inverse of the matrix
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
0 & 1 & 1
\end{bmatrix}
\].

II. Let \( A = \begin{bmatrix}
1 & 0 & 0 \\
2 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \) and \( B = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \).

(a) Show that \( A \) and \( B \) are row equivalent. Give a list of elementary matrices \( E_1, \ldots, E_k \) for which \( E_k \cdots E_1 A = B \).

(b) Explain why \( A \) and \( B \) cannot be column equivalent.

III. Let \( V \) be a vector space and let \( S = \{v_1, \ldots, v_k\} \) be a subset of \( V \). Recall that \( \text{span}(S) \) is the set of all linear combinations of element of \( S \), that is, \( \{\sum_{i=1}^k \lambda_i v_i \mid \lambda_i \in \mathbb{R}\} \). Verify that \( \text{span}(S) \) is a subspace of \( V \).

IV. Let \( W = \{at^2 + bt + c \mid c \geq 0\} \), that is, the set of all polynomials of degree at most 2 and having non-negative constant term. By giving a specific counterexample, show that \( W \) is not a subspace of \( \mathbb{P}_2 \) (the vector space of all polynomials of degree at most 2).

V. Let \( A \) be an \( m \times n \) matrix and consider the homogeneous system of linear equations given by \( AX = 0 \). Its solutions form a subset of \( \mathbb{R}^n \). Verify that the set of solutions is a subspace of \( \mathbb{R}^n \).

VI. Let \( V \) be a vector space and let \( S = \{v_1, \ldots, v_k\} \) be a subset of \( V \).

(a) Define what it means to say that \( S \) is linearly independent.

(b) Define what it means to say that \( S \) is a basis of \( V \).

(c) If \( V \) has dimension 6 and \( S \) is a subset consisting of five elements of \( V \), what can you say about \( S \), beyond just the fact that it is not a basis?

(d) If \( V \) has dimension 6 and \( S \) is a subset consisting of seven elements of \( V \), what can you say about \( S \), beyond just the fact that it is not a basis?

VII. Let \( V = \mathbb{R}_3 \) (the vector space of \( 1 \times 3 \) vectors), and let \( S = \left\{ \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \right\} \). Test \( S \) for linear independence. If it is not linearly independent, write one of its elements as a linear combination of the others.

VIII. If an \( n \times n \) nonsingular matrix \( A \) is equivalent to a matrix \( B \), then \( B \) must also be nonsingular. Why?

IX. If \( P \) is a nonsingular \( n \times n \) matrix, then its transpose \( P^T \) must also be nonsingular. Why?
X. Let $V$ be the vector space of all differentiable functions from the real numbers to the real numbers, with the usual addition and scalar multiplication operations.

(a) Verify that the subset $\{1, x, x^2, x^3\}$ is a linearly independent subset of $V$ (hint: suppose you have the zero function 0 written as a linear combination of these functions, then take derivatives three times).

(b) The same kind of argument as in (a) can be used to show that the set $\{1, x, x^2, x^3, x^4\}$ is a linearly independent subset of $V$, and even that the sets $\{1, x, x^2, x^3, \ldots, x^n\}$ are linearly independent for any choice of $n$ (do not try to check these facts). What does this tell us about the dimension of $V$. Why?