Mathematics 3333-001 Name (please print) ____________________

Examination I

February 19, 2009

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (12) Each of the following matrices is the augmented matrix of a system of linear equations, and is in row echelon form or reduced row echelon form. For each matrix, write a general expression for the solutions of the corresponding linear system, or else explain why the system is inconsistent. You may wish to simplify the matrix further before finding the solution.

1. $$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 = r$, $x_3 = s$, $x_5 = t$,
$x_4 = -x_5 + 5 = 5 - t$,
$x_1 = -x_2 + 7 = 7 - r$,
so the general solution is $(7 - r, r, s, 5 - t, t)$.

2. $$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 1$, $x_1 = -2x_2 = -2$, so $(-2, 1)$ is the unique solution.

3. $$\begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$ (hint: use row operations to create a lot of zeros before doing back substitution).

Using the row operations $R_1 - R_2 \longrightarrow R_1$, $R_2 - 2R_3 \longrightarrow R_2$, and $R_1 - R_2 \longrightarrow R_1$ (among various possible solutions), we simplify the matrix to

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}.$$

We then have $x_4 = r$, $x_3 = 2 - 2x_4 = 2 - 2r$, $x_2 = 2x_4 - 2 = 2r - 2$, and $x_1 = 2 - 2x_4 = 2 - 2r$, so the general solution is $(2 - 2r, 2r - 2, 2 - 2r, r)$.

4. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The general solution is (r, s, t, u).

II. (4) Simplify the following expressions, assuming that all matrices are $n \times n$ for a fixed size n.

1. $3I + A(3B - 4A^{-1})$

$$3I + A(3B - 4A^{-1}) = 3I + A(3B) - A(4A^{-1}) = 3I + 3AB - 4AA^{-1} = 3AB - I$$

2. $(AB^T - BA^T)^T$

$(AB^T - BA^T)^T = (AB^T)^T - (BA^T)^T = (B^T)^T A^T - (A^T)^T B^T = BA^T - AB^T$ (thus any matrix of the form $AB^T - BA^T$ is skew-symmetric)

III. (4) Write the following equation involving a linear combination of column vectors as a system of linear equations that has the same solutions:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{aligned} x_1 \phantom{{}+2x_2} + 3x_3 + x_4 &= 2 \\ 2x_1 + x_2 + 4x_3 + 3x_4 &= 5 \\ -x_1 + 2x_2 + 5x_3 + 4x_4 &= 8 \end{aligned}$$

IV. (6) (a) Suppose that A_1, $A_2, \ldots,$ A_k are nonsingular $n \times n$ matrices. Explain why the product $A_1 A_2 \cdots A_k$ is nonsingular.

We have $(A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}) \cdot A_1 A_2 \cdots A_k = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} \cdot I \cdot A_2 \cdots A_k = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} \cdot A_2 \cdots A_k = \cdots = I$, so $A_1 A_2 \cdots A_k$ is nonsingular with inverse $A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$.

(b) Give an example of nonzero 2×2 matrices A, B, and C for which $AB = AC$ but $B \neq C$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ but } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Show that if A, B, and C are 2×2 matrices with $AB = AC$, and A is nonsingular, then $B = C$.

Multiplying both sides of $AB = AC$ by A^{-1}, we would have $A^{-1}AB = A^{-1}AC$, so $IB = IC$ and $B = C$.

V. (5) Let $AX = B$ be a system of m linear equations in n variables, regarded as a matrix equation. Use properties of matrix operations to verify the following:

(a) If X_1 and X_2 are solutions, then $X_1 - X_2$ is a solution of the associated homogeneous system $AX = 0$.

$$A(X_1 - X_2) = AX_1 - AX_2 = B - B = 0.$$

(b) If X_1 and X_2 are solutions, then for any scalars r and s with $r + s = 1$, $rX_1 + sX_2$ is also a solution.

$$A(rX_1 + sX_2) = A(rX_1) + A(sX_2) = rAX_1 + sAX_2 = rB + sB = (r + s)B = B$$

VI. (4) For each of the following matrix transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, describe geometrically what the matrix transformation does to the plane.

1. $F(X) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X$

   This sends (x, y) to (y, x), which is reflection across the line $y = x$.

2. $F(X) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} X$

   This sends (x, y) to $(0, y)$, so each vector is horizontally projected to its "shadow" on the y-axis.

VII. (5) Let A be an $m \times n$ matrix, $A = [a_{ij}]$. Let I be the $n \times n$ identity matrix. By calculating the (i, j) entry of the product $A\,I$, show that $A\,I = A$.

Denoting the (i, j) entry of a matrix M by $M_{i,j}$, we have

$$\begin{aligned}(A\,I)_{i,j} &= \sum_{k=1}^{n} A_{i,k} I_{k,j} = A_{i,1}I_{1,j} + A_{i,2}I_{2,j} + \cdots + A_{i,j}I_{j,j} + A_{i,j+1}I_{j+1,j} + \cdots A_{i,n}I_{n,j} \\ &= A_{i,1} \cdot 0 + A_{i,2} \cdot 0 + \cdots + A_{i,j-1} \cdot 0 + A_{i,j} \cdot 1 + A_{i,j+1} \cdot 0 + \cdots A_{i,n} \cdot 0 = A_{i,j}\ ,\end{aligned}$$

so $A\,I = A$.

VIII. (4) The inverse of a certain 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find all solutions of the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 2 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 1 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= -1 \end{aligned}$$

The system can be regarded as $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, so the unique solution is $A^{-1} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$.

IX. (6) For the following system of linear equations involving numbers a, b, and c:

$$\begin{aligned} 2x - 5y + 3z &= a \\ 3x - 8y + 5z &= b \\ x - 3y + 2z &= c \end{aligned}$$

(a) Find a condition on a, b, and c so that the system is consistent for any choice of values of a, b, and c that satisfy the condition.

Writing the augmented matrix and using Gauss-Jordan elimination, we calculate:

$$\begin{bmatrix} 2 & -5 & 3 & a \\ 3 & -8 & 5 & b \\ 1 & -3 & 2 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & -1 & a-2c \\ 0 & 1 & -1 & b-3c \\ 1 & -3 & 2 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 2 & c \\ 0 & 1 & -1 & a-2c \\ 0 & 0 & 0 & -a+b-c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 3a-5c \\ 0 & 1 & -1 & a-2c \\ 0 & 0 & 0 & a-b+c \end{bmatrix}$$

The condition is that $a - b + c = 0$.

(b) Assuming that the condition is satisfied, obtain an expression (which will involve some of a, b, or c) for the general solution.

Assuming that $a - b + c = 0$, so that the last row is zero, we have $x_3 = r$, $x_2 = x_3 + a - 2c = r + a - 2c$, and $x_1 = x_3 + 3a - 5c = r + 3a - 5c$, so the general solution is $(r + 3a - 5c, r + a - 2c, r)$.