Mathematics 3333-001 Name (please print) ______________________

Examination I

February 19, 2009

Instructions: Give concise answers, but clearly indicate your reasoning.

**I.** (12) Each of the following matrices is the augmented matrix of a system of linear equations, and is in row echelon form or reduced row echelon form. For each matrix, write a general expression for the solutions of the corresponding linear system, or else explain why the system is inconsistent. You may wish to simplify the matrix further before finding the solution.

1. $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$ (hint: use row operations to create a lot of zeros before doing back substitution).

4. $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**II.** (4) Simplify the following expressions, assuming that all matrices are $n \times n$ for a fixed size $n$.

1. $3I + A(3B - 4A^{-1})$
2. $(AB^T - BA^T)^T$

**III.** (4) Write the following equation involving a linear combination of column vectors as a system of linear equations that has the same solutions:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

**IV.** (6) (a) Suppose that $A_1, A_2, \dots, A_k$ are nonsingular $n \times n$ matrices. Explain why the product $A_1 A_2 \cdots A_k$ is nonsingular.

(b) Give an example of nonzero $2 \times 2$ matrices $A$, $B$, and $C$ for which $AB = AC$ but $B \neq C$.

(c) Show that if $A$, $B$, and $C$ are $2 \times 2$ matrices with $AB = AC$, and $A$ is nonsingular, then $B = C$.

**V.** (5) Let $AX = B$ be a system of $m$ linear equations in $n$ variables, regarded as a matrix equation. Use properties of matrix operations to verify the following:

(a) If $X_1$ and $X_2$ are solutions, then $X_1 - X_2$ is a solution of the associated homogeneous system $AX = 0$.

(b) If $X_1$ and $X_2$ are solutions, then for any scalars $r$ and $s$ with $r + s = 1$, $rX_1 + sX_2$ is also a solution.

**VI.** (4) For each of the following matrix transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, describe geometrically what the matrix transformation does to the plane.

1. $F(X) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X$

2. $F(X) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} X$

**VII.** (5) Let $A$ be an $m \times n$ matrix, $A = [a_{ij}]$. Let $I$ be the $n \times n$ identity matrix. By calculating the $(i, j)$ entry of the product $AI$, show that $AI = A$.

**VIII.** (4) The inverse of a certain $3 \times 3$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find all solutions of the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 2 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 1 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= -1 \end{aligned}$$

**IX.** For the following system of linear equations involving numbers $a$, $b$, and $c$:
(6)

$$\begin{aligned} 2x - 5y + 3z &= a \\ 3x - 8y + 5z &= b \\ x - 3y + 2z &= c \end{aligned}$$

(a) Find a condition on $a$, $b$, and $c$ so that the system is consistent for any choice of values of $a$, $b$, and $c$ that satisfy the condition.

(b) Assuming that the condition is satisfied, obtain an expression (which will involve some of $a$, $b$, or $c$) for the general solution.