Mathematics 3333-001 Name (please print) ______________________

Final Examination

May 13, 2009

Instructions: Give concise answers, but clearly indicate your reasoning.

I. (9) (a) Suppose that $A_1, A_2, \ldots, A_k$ are nonsingular $n \times n$ matrices. Explain why the product $A_1 A_2 \cdots A_k$ is nonsingular.

We have $(A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}) \cdot A_1 A_2 \cdots A_k = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} \cdot I \cdot A_2 \cdots A_k = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} \cdot A_2 \cdots A_k = \cdots = I$, so $A_1 A_2 \cdots A_k$ is nonsingular with inverse $A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$.

(Alternatively, we have $\det(A_1 A_2 \cdots A_k) = \det(A_1)\det(A_2) \cdots \det(A_k) \neq 0$, so $A_1 A_2 \cdots A_k$ is nonsingular.)

(b) Give an example of nonzero 2×2 matrices A, B, and C for which $AB = AC$ but $B \neq C$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ but } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) Show that if A, B, and C are 2×2 matrices with $AB = AC$, and A is nonsingular, then $B = C$.

Multiplying both sides of $AB = AC$ by A^{-1}, we would have $A^{-1}AB = A^{-1}AC$, so $IB = IC$ and $B = C$.

II. (3) As you know, a *matrix transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function of the form $F(X) = AX$, where A is an $m \times n$ matrix and a point X in $\mathbb{R}^n$ is regarded as an $n \times 1$ column vector. Verify (using properties of matrix addition, matrix multiplication, and scalar multiplication) that any matrix function F must be linear.

$$F(aX + bY) = A(aX + bY) = A(aX) + A(bY) = aAX + bAY = aF(X) + bF(Y)$$

III. (7) Let $L \colon V \to W$ be a linear transformation.

(a) Define the *kernel* of L. Verify that it is a subspace of V.

The kernel of L is the set of elements x in V such that $L(x) = 0$.

Suppose that x_1 and x_2 are in the kernel of L, so $L(x_1) = 0$ and $L(x_2) = 0$. We have $L(x_1 + x_2) = L(x_1) + L(x_2) = 0 + 0 = 0$, so $x_1 + x_2$ is also in the kernel, and for any scalar λ, $L(\lambda x_1) = \lambda L(x_1) = \lambda \cdot 0 = 0$, so λx_1 is in the kernel.

(b) Define the *range* of L (it is a subspace of W, but you do not need to verify this).

The range of L is the set of elements y in W such that $L(x) = y$ for some x in V.

IV. (4) Let V be a vector space and let $S = \{v_1, \ldots, v_k\}$ be a subset of V. Verify that $\operatorname{span}(S)$ is a subspace of V.

Suppose that $a_1 v_1 + \cdots + a_k v_k$ and $b_1 v_1 + \cdots + b_k v_k$ are in $\operatorname{span}(S)$. Then

$$(a_1 v_1 + \cdots + a_k v_k) + (b_1 v_1 + \cdots + b_k v_k) = (a_1 + b_1) v_1 + \cdots + (a_k + b_k) v_k \ ,$$

which is also in $\operatorname{span}(S)$. Now, suppose that $\lambda \in \mathbb{R}$. Then

$$\lambda(a_1 v_1 + \cdots + a_k v_k) = (\lambda a_1) v_1 + \cdots + (\lambda a_k) v_k \ ,$$

which is also in $\operatorname{span}(S)$.

V. A certain matrix A has 15 rows and 20 columns.
(6)

(a) What rank must A have in order that its null space have dimension 8? Explain why.

It must have rank 12. For the rank plus the nullity must equal the number of columns, giving rank plus 8 equal 20.

(b) What nullity must A have in order that the nonhomogeneous system $AX = B$ have at least one solution for every choice of B? Explain why.

The nullity must be 5. Since the possible B lie in $\mathbb{R}^{15}$, we are asking what condition on the nullity of A will make the column rank equal to 15. Since column rank equals rank, we want 15 plus the nullity to equal 20.

VI. A certain matrix transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ is given by multiplication by a 3×3 matrix A, of nullity
(5) 1. Draw our standard picture of two 3-dimensional spaces, representing the domain and codomain of the matrix transformation, showing a possible null space, row space, and column space for A (label which is which, in your picture).

(See the classic picture found in our rank and nullity handout.)

VII. Let A be the 4×4 matrix $\begin{bmatrix} t & 0 & 0 & 1 \\ 0 & 0 & t & 0 \\ 0 & t & 0 & 0 \\ 1 & 0 & 0 & t \end{bmatrix}$, which depends on the value of the variable t. Use the *cofactor*
(4)
expansion method, expanding *across the second row,* to calculate the determinant of A.

$$\begin{vmatrix} t & 0 & 0 & 1 \\ 0 & 0 & t & 0 \\ 0 & t & 0 & 0 \\ 1 & 0 & 0 & t \end{vmatrix} = t \cdot (-1) \cdot \begin{vmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{vmatrix} = -t \cdot t \begin{vmatrix} t & 1 \\ 1 & t \end{vmatrix} = t^2(1 - t^2)$$

where the second equality comes from expanding across the second row.

VIII. Let $A = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ t & 0 & 1 \end{bmatrix}$, for which $\det(A) = 1 - t^2$. Assuming that $t \neq 1$ and $t \neq -1$, so that A is nonsingular,
(5)
use the row operation method to compute A^{-1}. (Hint: Start with the elementary row operation $R_3 - tR_1 \to R_3$.)

$$\begin{bmatrix} 1 & 0 & t & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ t & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & t & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-t^2 & -t & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{1-t^2} & 0 & \frac{-t}{1-t^2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{-t}{1-t^2} & 0 & \frac{1}{1-t^2} \end{bmatrix}, \text{ so } A^{-1} = \begin{bmatrix} \frac{1}{1-t^2} & 0 & \frac{-t}{1-t^2} \\ 0 & 1 & 0 \\ \frac{-t}{1-t^2} & 0 & \frac{1}{1-t^2} \end{bmatrix}$$

(Going from the second matrix to the third is $R_1 - \frac{t}{1-t^2} R_3 \to R_1$, followed by some algebraic simplification.)

IX. Let u and v be vectors in an inner product space V.
(5)
(a) Use properties of the inner product to determine how $\| u+v \|^2$ is related to $\| u \|^2 + \| v \|^2$. (Hint: $\| u+v \|^2 = (u+v, u+v)$.)

$$\| u+v \|^2 = (u+v, u+v) = (u, u+v) + (v, u+v) = (u,u) + (u,v) + (v,u) + (v+v) = \| u \|^2 + 2(u,v) + \| v \|^2$$

(b) Deduce that if u and v are orthogonal for the inner product, then $\| u+v \|^2 = \| u \|^2 + \| v \|^2$.

When u and v are orthogonal, $(u,v) = 0$, so the formula in part (a) becomes simply $\| u+v \|^2 = \| u \|^2 + \| v \|^2$.

X. Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in a vector space V.
(6)
(a) Define what it means to say that S is *linearly independent.*

S is linearly independent if $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$ only when all $\lambda_i = 0$.
(Alternatively, one can define S to be linearly dependent if there are numbers $\lambda_1, \ldots, \lambda_n$, not all zero, so that $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$, and then S is linearly independent if it is not linearly dependent.)

(b) Show that if S is linearly independent, and

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n \ ,$$

then $a_i = b_i$ for all i. (Hint: subtract $b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$ from both sides of the equation.)

Subtracting $b_1 v_1 + b_2 v_2 + \cdots + b_n v_n$ from both sides gives

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n - (b_1 v_1 + b_2 v_2 + \cdots + b_n v_n) = 0$$
$$(a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \cdots + (a_n - b_n) v_n = 0$$

Since S is linearly independent, we must have every $a_i - b_i = 0$, that is, every $a_i = b_i$.

XI. (4) Find the characteristic polynomial of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}$.

Expanding down the first column, we have $\begin{vmatrix} \lambda-1 & -2 & -1 \\ 0 & \lambda-1 & -2 \\ 0 & -3 & \lambda-2 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda-2 \end{vmatrix}$
$= (\lambda-1)\big((\lambda-1)(\lambda-2)-6\big) = (\lambda-1)(\lambda^2-3\lambda-4) = (\lambda-1)(\lambda+1)(\lambda-4)$.

XII. (6) The eigenvalues of the matrix $A = \begin{bmatrix} -2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ are -2, 1, and 4. Find an eigenvector associated to -2, and an eigenvector associated to 1.

For $\lambda = -2$, we have $\lambda I_3 - A = \begin{bmatrix} 0 & 2 & -3 \\ 0 & -5 & 2 \\ 0 & 1 & -4 \end{bmatrix}$ which becomes $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ in row echelon form. The null space is $\left\{\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}\right\}$, so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector associated to -2.

For $\lambda = 1$, we have $\lambda I_3 - A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ which becomes $\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ in row echelon form. The null space is $\left\{\begin{bmatrix} r/3 \\ r \\ r \end{bmatrix}\right\}$, so $\begin{bmatrix} 1/3 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector associated to 1.

XIII. Find a basis for the eigenspace associated with $\lambda = 3$ for the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$.
(5)

We have $3I_3 - A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ -2 & 0 & -2 \end{bmatrix}$ which becomes $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in row echelon form. The null space is $\left\{ \begin{bmatrix} -s \\ r \\ s \end{bmatrix} \right\}$,

so a basis for the null space is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

XIV. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$ the eigenvalues are -1, 1, and 4, and associated eigenvectors are $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$,
(6)

$\begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (you do not need to calculate these or check them).

(a) Write down a diagonal matrix D that is similar to A.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) Write down a matrix P so that $P^{-1}AP = D$.

The matrix P^{-1} is the matrix for the change of basis from the standard basis to the basis $\left\{ \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, so $P = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -6 & 0 \\ 2 & 4 & 1 \end{bmatrix}$.