in $\mathcal{P}[0,\infty)^{\mathcal{C}}$ is a closed ball of dimension $6g-6+2p$ whose interior is $\mathcal{P}\circ\mathcal{L}(\mathcal{T})$. The boundary points of this ball are usually described in one of two ways: using measured geodesic laminations, or using measured foliations. We shall adopt the latter method, which is better designed for working with Teichmüller geodesics. However, we shall extend the class of measured foliations to include "partial measured foliations", which have some of the flexibility of geodesic laminations.

2.3 The mapping class group

Let $\mathrm{Homeo}(S)$ be the group of homeomorphisms of S, and let $\mathrm{Homeo}_0(S)$ be the normal subgroup of homeomorphisms isotopic to the identity on S. The *mapping class group* is $\mathcal{MCG} = \mathcal{MCG}(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S)$. There is a natural action of $\mathcal{MCG}$ on the Teichmüller space $\mathcal{T}$. Indeed, as we define other natural topological and geometric structures on surfaces on which $\mathrm{Homeo}(S)$ acts, the action of $\mathrm{Homeo}_0(S)$ induces an equivalence relation on such structures which coincides with isotopy, and $\mathcal{MCG}$ acts naturally on the set of isotopy classes. Examples include the actions of $\mathcal{MCG}$ on the isotopy classes of measured foliations $\mathcal{MF}$, and on the isotopy classes of quadratic differentials QD.

The action of $\mathcal{MCG}$ on $\mathcal{T}$ is properly discontinuous, and the quotient space is naturally a smooth orbifold called the *moduli space* of S, denoted $\mathcal{M} = \mathcal{M}(S) = \mathcal{T}/\mathcal{MCG}$.

2.4 Quadratic differentials and their horizontal and vertical foliations

The singularities of measured foliations are locally modelled on singularities of horizontal foliations of quadratic differentials. We shall therefore pause for a brief foray into quadratic differentials, which will be covered in more detail later.

If we give S a conformal structure, a *meromorphic quadratic differential* q on S is an expression $q = f(z)\,dz^2$ for each local coordinate z, such that $f(z)$ is meromorphic, and such that for any overlap map $z' \mapsto z$ between local coordinates, if $q = f(z)\,dz^2$ in the z coordinate, then in the z' coordinate we have $q = f(z(z'))\left(\frac{dz}{dz'}\right)^2 dz'^2$. Zeroes and poles of q are well-defined, as is the order of a zero or pole. q determines an area form, expressed in a coordinate z as $|f(z)|\,|dz|^2$, and the total area is denoted $\|q\|$. Note that q is integrable, meaning that $\|q\|$ is finite, if and only if q has a meromorphic extension to the filled in surface $\bar{S}$ so that each pole has degree 1. In studying the Teichmüller space of a Riemann surface S of finite type, the most relevant quadratic differentials on S are those which are holomorphic and integrable. For the rest of the paper, therefore, a *quadratic differential* on S is assumed to have these properties.

Consider a quadratic differential q. Near any regular point p there is a local coordinate z so that $q = dz^2$; the germ of this coordinate is unique up to transformations of the form $z \mapsto \pm z + c$. Near any singular point p there is a local coordinate z, taking p to the origin of the z-plane, in which $q = z^n\, dz^2$, such that either $n = -1$ and p is a pole of q, or $n \geq 1$ and p is a zero of order n; the coordinate z is unique up to multiplication by an $(n+2)^{\text{nd}}$ root of unity (completely unique in the case $n = -1$). In any of these cases we refer to z as a *canonical coordinate* for q near p.

A quadratic differential q determines two singular transversely measured foliations, called the *horizontal* and *vertical* foliations, denoted $\mathcal{F}^h(q)$ and $\mathcal{F}^v(q)$. These foliations have the same singular set, namely the set of zeroes and poles of q. Near a nonsingular point p with canonical coordinate $z = x + iy$, horizontal leaf segments are parallel to the x-axis and the transverse measure on $\mathcal{F}^h(q)$ is defined by integration of $|dy|$, while vertical leaf segments are parallel to the y-axis with transverse measure defined by integrating $|dx|$. These measured foliations are well-defined, because the canonical coordinate near a regular point is well defined up to transformations of the form $z \mapsto \pm z + c$.

To visualize the measured foliations $\mathcal{F}^h(q)$ and $\mathcal{F}^v(q)$ near a singular point p with canonical coordinate z, we have $q = z^{k-2}\, dz^2$ for some $k \geq 1$; either $k \geq 3$ and p is a zero of order $k-2$, or $k = 1$ and p is a pole of order 1. Near any $z \neq 0$ an easy calculation shows that $z' = z^{k/2}$ is a canonical coordinate for $z^{k-1}\, dz^2$, using either choice of the square root in the case where k is odd. Horizontal and vertical leaves of $z^{k-2}\, dz^2$ can therefore be described as follows. For each $i \in \mathbf{Z}/k\mathbf{Z}$ let S_i be the angular sector of $\mathbf{C}$ defined in polar coordinates by $2\pi i/n \leq \theta \leq 2\pi(i+1)/n$. Let $z' = x' + iy'$ vary over the closed upper half plane of $\mathbf{C}$, and let $z'^{2/k}$ denote the $2/k$th root of z' taking values in the angular sector S_0. For each $i \in \mathbf{Z}/k\mathbf{Z}$, the map $z' \xrightarrow{\phi_i} z = \exp(2\pi i/k) z'^{2/k}$ takes values in the sector S_i, the horizontal leaves of $z^{k-2}\, dz^2$ in S_i are the images under the map ϕ_i of lines parallel to the x'-axis in the closed upper half z' plane, and the transverse measure is obtained by pushing forward $|dy'|$; a similar discussion holds for vertical leaves. The ray $R_i = S_{i-1} \cap S_i$, whose angle is $\theta = 2\pi i/n$, is a leaf of the horizontal foliation called a *separatrix attached to the origin*. The set of separatrices, indexed by $\mathbf{Z}/k\mathbf{Z}$, has a natural circular ordering. By convention, a regular point of a foliation will sometimes be called a 2-*pronged singularity*, especially when that point is a puncture on a surface of finite type.

2.5 Measured foliations

The general concept of a measured foliation on a surface of finite type is obtained by using the horizontal measured foliation of a quadratic differential as a local model.

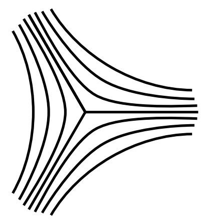

Figure 3: A k-pronged singularity ($k = 3$), modelled on the horizontal foliation of $z^{k-2}\,dz^2$ near the origin of $\mathbf{C}$. Leaf segments are pre-images, under a transformation $z' = z^{k/2}$ (with any choice of the root), of horizontal segments in the z' plane. Each ray passing through a k^{th} root of unity is a leaf, called a *separatrix*. The region between two separatrices is foliated like the horizontal foliation of the upper half-plane.

For surfaces with boundary, we need two other local models to define measured foliations tangent to the boundary: a *regular tangential boundary point* is locally modelled on the horizontal foliation of the closed upper half plane of $\mathbf{C}$, near the origin; and a *k-pronged tangential boundary singularity* with $k \geq 3$ is locally modelled on the horizontal foliation of $z^{2k-2}\,dz^2$ near the origin in the closed upper half plane of $\mathbf{C}$, with 2 separatrices on the boundary and $k-2$ separatrices pointing into the interior.

Consider now a finite type surface-with-boundary $F = \bar{F} - P(F)$, meaning that $\bar{F}$ is a compact, oriented surface-with-boundary and $P(F)$ is a finite subset of $\text{int}(F)$. A *measured foliation* on F is a foliation $\mathcal{F}$ on $\bar{F}$ with finitely many prong singularities $\text{sing}(\mathcal{F})$ and with a positive transverse Borel measure, such that each singularity of $\mathcal{F}$ in $\text{int}(\bar{F}) - P$ is a k-pronged singularity for some $k \geq 3$, each puncture is a k-pronged singularity for some $k \geq 1$, and each point of ∂F is either a regular tangential boundary point or a k-pronged tangential boundary singularity for some $k \geq 3$. By convention every puncture of $\bar{F}$ is considered to be a singularity, even if it is a 2-pronged singularity, so $P \subset \text{sing}(\mathcal{F})$. The restriction $\mathcal{F} \bigm| F - \text{sing}(\mathcal{F})$ is a true foliation; it is locally modelled on the horizontal foliation of $\mathbf{R}^2$ with overlap maps of the form $(x, y) \mapsto (f(x, y), \pm y + c)$, and so the transverse measure is defined in local models as integration of $|dy|$. The interested reader can work out the form of overlap maps between regular and singular models for $\mathcal{F}$, and between two singular models.

Given a leaf ℓ of $\mathcal{F} \bigm| \bar{F} - \text{sing}(\mathcal{F})$, let $\overline{\ell}$ be obtained from ℓ by adding any singularity of $\mathcal{F}$ which is the limit point of some end of ℓ; we define $\overline{\ell}$ to be a *nonsingular leaf of $\mathcal{F}$*. There are several types of nonsingular leaves: a *bi-infinite* leaf is homeomorphic to $\mathbf{R}$; a *closed leaf* is a circle containing no singularities; an *infinite separatrix* is homeomorphic to $[0,\infty)$ with one endpoint at a singularity; and a *saddle connection* is a compact leaf containing a singularity, either an arc with endpoints at distinct singularities, or a circle containing one singularity. A *nonsingular leaf segment* is a compact segment contained in a nonsingular leaf.

We still have not defined what we mean by a leaf of $\mathcal{F}$, unadorned with any qualifiers. Consider a compact segment α which is a union of nonsingular leaf segments. Given a transverse orientation V on α, we say that α is *perturbable* in the direction of V if there exists an embedding $f\colon \alpha\times[0,1]\to\bar{F}$ such that $f(\alpha\times 0)=\alpha$, the oriented segment $f(x\times[0,1])$ represents the transverse orientation V at the point $f(x)\in\alpha$, and for each $t\in(0,1]$ the set $f(\alpha\times t)$ is a nonsingular leaf segment. We say that α is a *leaf segment* if it has a perturbable transverse orientation. For example, if α is a nonsingular leaf segment then both transverse orientations are perturbable and in particular α is a leaf segment. More generally, perturbability means that at a singularity $s\in\operatorname{int}(\alpha)$, α contains two separatrices incident to the same sector at s, and V points into that sector. Finally, a *leaf* ℓ of F is the embedded image of either $\mathbf{R}$ or S^1, whose image is a union of nonsingular leaves, such that for some transverse orientation V on ℓ, the restriction of V to every compact subsegment of ℓ is perturbable. Notice that if $\ell\approx\mathbf{R}$ then ℓ is *never* globally perturbable in either direction; this uses the existence of a positive transverse measure on $\mathcal{F}$.

A *finite separatrix* of $\mathcal{F}$ is a nonsingular leaf segment ℓ with one end at a singularity $s\in\operatorname{sing}(\mathcal{F})$. More precisely we say that ℓ is a *separatrix located at* s. If s is an n-pronged singularity then there are n distinct separatrix germs at s. Each infinite separatrix represents a unique separatrix germ, each saddle connection represents exactly two separatrix germs, and every separatrix germ is uniquely represented either by an infinite separatrix or by a saddle connection.

A *leaf cycle* of $\mathcal{F}$ is a union of saddle connections which forms either an embedded circle in $\bar{F}$, or an embedded arc in $\bar{F}$ not lying in ∂F each of whose endpoints is either a puncture or a boundary singularity. Note that if $\mathcal{F}$ has a closed leaf then $\mathcal{F}$ has a leaf cycle; the converse is not true, because a leaf cycle need not have a perturbable transverse orientation.

Suppose that $\mathcal{F}$ is a measured foliation on F and α is a saddle connection which is not a leaf cycle (so α is an embedded arc, and either $\alpha\subset\partial F$ or at least one endpoint is neither a puncture nor a boundary singularity). Collapsing α to a point induces a new measured foliation on F which we say is obtained from $\mathcal{F}$ by a *Whitehead collapse*. The equivalence relation on the set of measured foliations on F generated by Whitehead collapse and isotopy is called *Whitehead equivalence*.

The term "measured foliation" without any other qualifiers will always imply, as defined above, that each boundary point is a regular tangential boundary point or a tangential boundary singularity. We shall have a limited need for *boundary transverse measured foliations* of F as well. These have the same local models in

$F - \partial F$, but on the boundary the local models for *regular* and *k-pronged transverse* boundary points are the vertical foliations of dz^2 and $z^{2k-2}\,dz^2$, respectively, restricted to the closed upper half plane, near the origin. The only place where we make significant use of boundary transverse foliations is in the tie bundle over a train track, in Section 3.

The definition of a leaf of $\mathcal{F}$ that we have adopted in this section behaves well under Whitehead equivalence. That is, if $\mathcal{F}, \mathcal{F}'$ are Whitehead equivalent measured foliations then there is a natural bijection between leaves of $\mathcal{F}$ and leaves of $\mathcal{F}'$. This is obvious when $\mathcal{F}, \mathcal{F}'$ are isotopic. It is not hard to check for a Whitehead collapse from $\mathcal{F}$ to $\mathcal{F}'$: it follows immediately from the observation that if α is a segment consisting of a union of leaf segments of $\mathcal{F}$, and if α' is the image of α, then α' is a union of leaf segments of $\mathcal{F}'$, and α has a perturbable transverse orientation if and only if α' does.

Also, the definition of a leaf of $\mathcal{F}$ is consistent with the concept of a leaf of an equivalent measured geodesic lamination λ, in the following sense. Recall that equivalence means that there exists a map $\pi\colon (S, \lambda) \to (S, \mathcal{F})$ that collapses components of $S - \lambda$ to leaf cycles of $\mathcal{F}$, takes leaf segments of λ to leaf cycles of $\mathcal{F}$, and pushes the transverse measure on λ forward to the transverse measure on $\mathcal{F}$. Under these circumstances, π take a leaf segment of λ to segment which is a union of leaf segments of $\mathcal{F}$ and which has a perturbable transverse orientation, and hence π takes leaves of λ bijectively to leaves of $\mathcal{F}$.

Arational measured foliations. A measured foliation $\mathcal{F}$ on F is *arational* if every leaf cycle is a component of ∂F. Equivalently, for every essential, nonperipheral simple closed curve c, the intersection number of $\mathcal{F}$ with c is nonzero. Arationality is invariant under Whitehead equivalence.

In the case where F has no punctures and no boundary, arationality of $\mathcal{F}$ means that there are no leaf cycles at all; equivalently, the union of saddle connections of $\mathcal{F}$ is a disjoint union of trees. When F has punctures, arationality of $\mathcal{F}$ means that the union of saddle connections is a disjoint union of trees, *and* each of these trees contains at most one puncture. When F has punctures and boundary, arationality of $\mathcal{F}$ means that for each component C of the union of saddle connections, either C is a tree containing at most one puncture, or C contains a component of ∂F as a deformation retract and C contains no puncture at all.

A property related to arationality is *minimality*, which means that each leaf of $\mathcal{F}$ is dense. Each arational measured foliation $\mathcal{F}$ is minimal: if there is a nondense leaf ℓ then its closure is a proper 2-complex in S, whose frontier is a union of leaf segments of $\mathcal{F}$. On a torus with ≤ 1 puncture the converse is true: the arational measured foliations are the same as the minimal measured foliations, and they are