Math 6843 homework

11. (2/21) Let α and β be two curves on $F = F_{2,0}$ which intersect in two points with the same orientations. Let G be a manifold regular neighborhood of $\alpha \cup \beta$, like the neighborhoods considered in the proof that $d(\alpha, \beta) \leq 2 \log_2(i(\alpha, \beta)) + 2$, so that G deformation retracts to $\alpha \cup \beta$.

    1. Use the classification of surfaces to verify that G must be a twice-punctured torus $F_{1,2}$. Hint: Show that G has $\chi(G) = -2$, $g \geq 1$, and $b \geq 1$.
    2. Since G is a twice-punctured torus, $\overline{S - G}$ is either an annulus or the union of a once-punctured torus and a disk. On a standard picture of $F_{2,0}$, try to draw an example of each type.

12. (2/21) For a (compact, connected, orientable) surface S with nonempty boundary, a pair of curves α and β is said to *fill* S if every nontrivial curve in S must meet $\alpha \cup \beta$. Equivalently, every component of $S - (\alpha \cup \beta)$ is either an open disk or a half-open annulus ($[0, 1) \times S^1$) that contains a boundary circle of S. Try to draw two curves that fill the twice-punctured torus $F_{1,2}$.

13. (2/28) Let A be a 2×2 matrix with complex entries.

    1. Check that the characteristic polynomial of A is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$.
    2. Use the Cayley-Hamilton Theorem to deduce that if $A \in \mathrm{SL}(2, \mathbb{R})$ and $\mathrm{tr}(A) = -1$, then $A^3 - I = 0$, and hence A has order 3. Obtain similar results when $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(A) = 1$.
    3. Show that if $A \in \mathrm{SL}(2, \mathbb{R})$ and $|\mathrm{tr}(A)| > 2$, then A has two real eigenvalues which are reciprocals, one of which has absolute value greater than 1. Deduce that A has infinite order.

14. (2/28) Following the program we used with $\begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$, analyze the action of $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ on the torus. That is:

1. Calculate the eigenvalues $\{\lambda, 1/\lambda\}$ and associated length 1 eigenvectors v_λ and $v_{1/\lambda}$.
2. Express the standard basis $\{e_1, e_2\}$ (that correspond to the curves we call L and M on the torus) in terms of the basis $\{v_\lambda, v_{1/\lambda}\}$. Use this to redraw the standard fundamental domain (spanned by e_1 and e_2) using the basis $\{v_\lambda, v_{1/\lambda}\}$.
3. Examine the stable and unstable foliations, first from the viewpoint of the basis $\{v_\lambda, v_{1/\lambda}\}$. Find their slopes with respect to the standard basis, and try to imagine them on the torus and how the element $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ acts on the torus.