Math 6843 homework

1. (due 1/22) Let $M$ be the Möbius band.
   (a) Construct an explicit (in coordinates) deformation retraction from $M$ onto its center circle.
   (b) Prove that $M$ does not retract to $\partial M$.

2. (1/22) Let $M$ be the Möbius band. Regard its boundary as $S^1$. Using our definitions, determine the quotient space $M/\langle x \sim y \text{ for all } x, y \in \partial M \rangle$. Give at least a couple of different explanations, for example, one using Euler characteristic and one describing a homeomorphism to one of the manifolds that we discussed in class.

3. (1/22) Let $M$ be the Möbius band. Regard its boundary as $S^1$. Using our definitions, determine the quotient space $M/\langle x \sim -x \text{ for all } x \in \partial M \rangle$. Give at least a couple of different explanations, for example, one using Euler characteristic and one describing a homeomorphism to one of the manifolds that we discussed in class.

4. (1/24) (a) Regard the punctured Möbius band $F$ as two disks connected by three bands, one of them twisted. Find a loop in the interior of $F$ that bounds a crosscap.
   (b) Regard the punctured Klein bottle $F$ as two disks connected by three bands, two of them twisted. Find two loops in the interior of $F$ that bound disjoint crosscaps.

5. (1/24) Draw a punctured torus imbedded in $\mathbb{R}^3$ so that its boundary circle is a figure-8 knot.

6. (1/29) Use the Classification Theorem to give an informal (i.e. mostly pictures) argument that every compact, connected orientable surface has an orientation-reversing self-homeomorphism. In fact, show the following:

   1. If $F$ has an even number of boundary circles, then it has an orientation-reversing involution (a homeomorphism $h$ for which $h \circ h$ is the identity) whose fixed-point set is a circle.
   2. If $F$ has an odd number of boundary circles, then it has an orientation-reversing involution whose fixed-point set is an arc.
   3. If $F$ has an even number of boundary circles, then it has an orientation-reversing involution whose fixed-point set is empty. [Rk: If an orientable surface $F$ has an odd number of boundary circles, then every involution $F$ has a fixed point. For if $F$ had a free involution $h$, the quotient map $F \to F/h$ would be a 2-covering map and hence $\chi(F/h) = \frac{1}{2}\chi(F)$, but $F$ has odd Euler characteristic so $\chi(F/h)$ would not be an integer.]