Mathematics 2443-003

Examination III

April 24, 2008

Instructions: Give brief answers, but clearly indicate your reasoning.

\[x = \rho \cos(\theta) \sin(\phi), \ y = \rho \sin(\theta) \sin(\phi), \ z = \rho \cos(\phi), \ dV = \rho^2 \sin(\phi) \ d\rho \ d\phi \ d\theta, \ \vec{r}_\phi \times \vec{r}_\theta = a \sin(\phi)(x\vec{i} + y\vec{j} + z\vec{k}),\]
\[
\|\vec{r}_\phi \times \vec{r}_\theta\| = a^2 \sin(\phi)
\]
\[dS = \sqrt{1 + g_x^2 + g_y^2} \ dD
\]
\[
\int_S (P \vec{i} + Q \vec{j} + R \vec{k}) \cdot d\vec{S} = \int_D -P \ g_x - Q \ g_y + R \ dD
\]
\[
\int_S \vec{F} \cdot d\vec{S} = \int_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \ dD
\]

I. For the following line integrals: write a definite integral, in terms of the specified parameter, whose value equals the value of the line integral, but do not evaluate the definite integral.

1. \[\int_C xy^2 \ ds, \] where \(C\) is parameterized by \(x = -t^2, \ y = t^3\) for \(1 \leq t \leq 2\).

2. \[\int_C (xy^2\vec{r}) \cdot d\vec{r}, \] where \(C\) is parameterized by \(x = -t^2, \ y = t^3\) for \(1 \leq t \leq 2\).

II. Let \(r\) and \(\theta\) be the usual polar coordinates on the plane. Calculate \(\frac{\partial \theta}{\partial x}\) as follows:

(i) Use implicit differentiation starting from \(r^2 = x^2 + y^2\) to calculate that \(\frac{\partial r}{\partial x} = \frac{x}{r}\).

(ii) Starting from \(x = r \cos(\theta)\), obtain an expression for \(\frac{\partial \theta}{\partial x}\) in terms of \(x\) and \(y\).

III. Find a normal vector to the surface given by the parameterization \(x = u^2, \ y = 2u \sin(v), \ z = u \cos(v)\) at the point corresponding to \((u, v) = (2, \pi/4)\).

IV. The figure to the right shows a vector field \(\vec{F} = P \vec{i} + Q \vec{j}\). Based on the probable behavior of the vector field, tell whether each of the following is zero, positive, negative, or cannot reasonably be determined from the information given: \(\frac{\partial P}{\partial x}, \ \frac{\partial Q}{\partial x}, \ \frac{\partial P}{\partial y}, \ \frac{\partial Q}{\partial y}, \ \text{div}(\vec{F}), \ \text{curl}(\vec{F}) \cdot \vec{k}\).

V. State the Fundamental Theorem for Line Integrals, and use it to evaluate \(\int_C (x^2\vec{i} + y\vec{j}) \cdot d\vec{r}\), where \(C\) is a path from \((1, 0)\) to \((2, 2)\).

VI. Calculate \(\int_C xe^{-x} \ dx + (x^3 + 3xy^2) \ dy\), where \(C\) is the unit circle with the clockwise orientation.
VII. Set up definite integrals whose values equal the values of the following surface integrals. Supply limits of integration, but do not carry out the evaluation of the definite integrals. 

1. \( \int_S y \, dS \) where \( S \) is the part of the paraboloid \( y = x^2 + z^2 \) that lies inside the cylinder \( x^2 + z^2 = 4 \). Express the definite integral in polar coordinates in the \( xz \)-plane, including specifying the limits of integration, but do not evaluate.

2. \( \int_S (x \hat{i} + x \hat{j} + 2z \hat{k}) \cdot d\vec{S} \), where \( S \) is the portion of the sphere \( x^2 + y^2 + z^2 = 4 \) with \( 0 \leq \phi \leq 3\pi/4 \), and with respect to the outward normal. Make use of the formulas given at the start of the exam; do not derive expressions for \( \vec{r}_\phi \) and \( \vec{r}_\theta \).

VIII. Let \( S \) be the sphere of radius \( a \) with center at the origin. The differential of surface area on \( S \) can be expressed in terms of \( d\phi \) and \( d\theta \). Using the picture shown to the right, explain why \( dS \) appears to be \( a^2 \sin(\phi) \, d\phi \, d\theta \).

IX. Let \( \vec{F}(x, y) \) be the vector field \( \frac{-y}{x^2 + y^2} \hat{i} + \frac{x}{x^2 + y^2} \hat{j} \). Let \( C_R \) be the circle of radius \( R \) centered at the origin of the \( xy \)-plane.

(i) Evaluate \( \int_{C_R} \vec{F} \cdot d\vec{r} \) by direct calculation using a parameterization of \( C_R \).

(ii) Let \( C \) be any simple (no self crossings) loop that encloses the origin. Give \( C \) the positive orientation. Let \( R \) be a number so large that \( C \) is entirely contained inside \( C_R \), and let \( D \) be the region lying between \( C_R \) and \( C \), so that the oriented boundary of \( D \) is \( C_R + (-C) \).

(a) Draw a sketch of a typical \( C \) and \( C_R \), showing the region \( D \).

(b) Use Green’s Theorem, which applies to the region \( D \) as long as we used its oriented boundary \( C_R + (-C) \), to calculate that \( \int_{C_R+(-C)} \vec{F} \cdot d\vec{r} = 0 \).

(c) What is the numerical value of \( \int_C \vec{F} \cdot d\vec{r} \)? Why?