Mathematics 2443-006H Name (please print) ____________________

Examination II

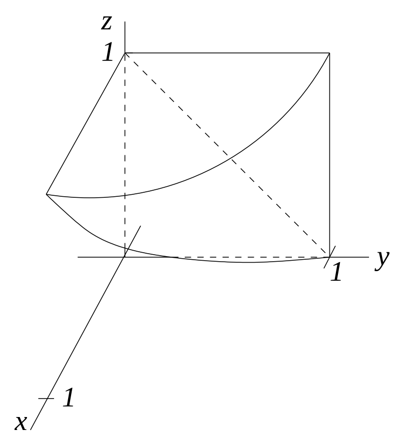
March 13, 2008

Instructions: Give brief, clear answers.

I. (6) Let a be a positive number and let T be the triangle in the xy-plane bounded by $x = 0$, $y = 0$, and $x + y = a$. Find the centroid $(\overline{x}, \overline{y})$ of T (i. e. the center of mass, assuming that $\rho = 1$) of T. You may take it as obvious that the centroid lies on the line $y = x$, so it is only necessary to calculate one of $\overline{x}$ or $\overline{y}$.

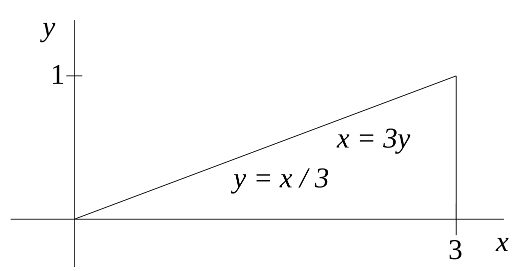
The mass m is the area of T, so $m = a^2/2$. We calculate $M_y = \iint_T x\,dA = \int_0^a \int_0^{a-x} x\,dy\,dx = \int_0^a xy\Big|_0^{a-x} dx = \int_0^a ax - x^2\,dx = \frac{ax^2}{2} - \frac{x^3}{3}\Big|_0^a = \frac{a^3}{6}$, so $\overline{x} = M_y/m = a/3$. Therefore the centroid is $(a/3, a/3)$.

II. (6) Let E be the region in the first octant bounded by the surfaces $x^2 + y^2 = 1$, $z = 1$, and $y + z = 1$ (so $z = 1$ forms the top of the solid). Sketch the region, and supply limits of integration, in cylindrical coordinates, for the integral $\iiint_E f(x, y, z)\,dV$.



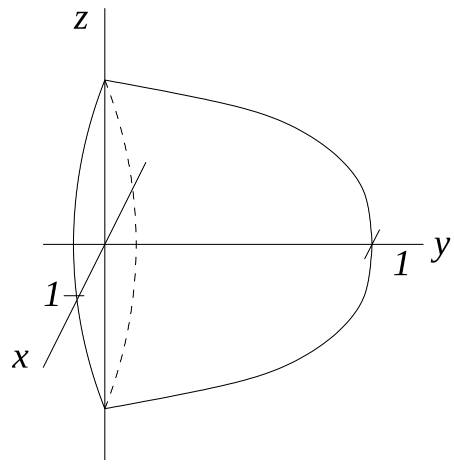
$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2} \int_0^1 \int_{1-r\sin(\theta)}^1 r\,f(r\cos(\theta), r\sin(\theta), z)\,dz\,dr\,d\theta\ .$$

III. (6) Evalute $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.



$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 x\,e^{x^2}/3\,dx = e^{x^2}/6\,\Big|_0^3 = (e^9 - 1)/6\ .$$

IV. (6) Find the surface area of the part of the paraboloid $y = 1 - x^2 - z^2$ that has $y \geq 0$.



$$dS = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dA = \sqrt{1 + 4x^2 + 4z^2}\, dA$$

$$\text{area}(S) = \iint_R \sqrt{1 + 4x^2 + 4z^2}\, dA = \int_0^{2\pi} \int_0^1 r\sqrt{1 + 4r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} (1 + 4r^2)^{3/2}/12 \Big|_0^1 \, d\theta = \int_0^{2\pi} (5\sqrt{5} - 1)/12 \; d\theta = \pi(5\sqrt{5} - 1)/6 \ .$$

V. (7) Find the z-coordinate $\overline{z}$ of the center of mass of the portion of the region E in the first octant that lies inside the sphere $x^2 + y^2 + z^2 = 4$, assuming that the density is proportional to the distance from the origin.

the density is $k\rho$

$$m = \iiint_E k\rho\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 k\rho^3 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} 4k \sin(\phi)\, d\phi\, d\theta = \int_0^{\pi/2} 4k\, d\theta = 2k\pi$$

$$M_{xy} = \iiint_E z\, k\rho\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 k\rho^4 \cos(\phi) \sin(\phi)\, d\rho\, d\phi\, d\theta$$

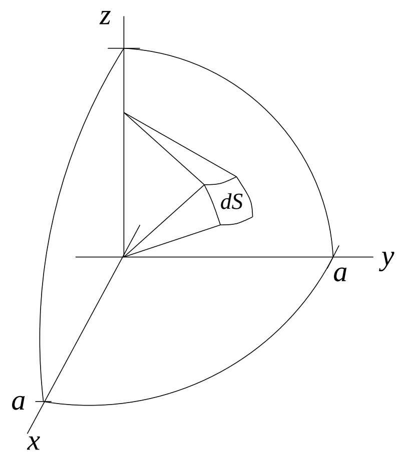
$$= \int_0^{\pi/2} \int_0^{\pi/2} 32k \cos(\phi) \sin(\phi)/5\, d\phi\, d\theta = \int_0^{\pi/2} 16k/5\, d\theta = 8k\pi/5$$

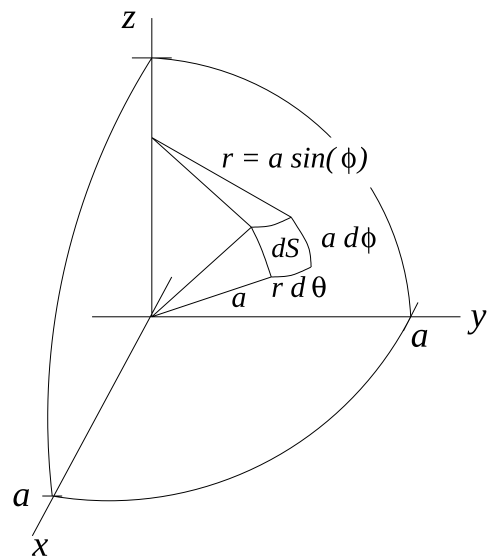
$$\overline{z} = M_{xy}/m = 4/5$$

VI. Let S be the sphere of radius a with center at the origin.
(6)

(a) The differential of surface area on S can be expressed in terms of $d\phi$ and $d\theta$. Using the picture shown to the right, explain why dS appears to be $a^2 \sin(\phi)\, d\phi\, d\theta$.

(b) Using the expression $dS = a^2 \sin(\phi)\, d\phi\, d\theta$, use a double integral in the variables ϕ and θ to calculate that the area of S is $4\pi a^2$.





$$\text{area}(S) = \iint_S dS = \int_0^{2\pi}\int_0^{\pi} a^2 \sin(\phi)\, d\phi\, d\theta = \int_0^{2\pi} -a^2\cos(\phi)\Big|_0^{\pi} d\theta = \int_0^{2\pi} 2a^2\, d\theta = 4\pi a^2 \ .$$

VII. Let $x = e^u \sin(t)$, $y = e^u \cos(t)$, and $z = f(x, y)$.
(5)

1. Calculate $\dfrac{\partial x}{\partial t}$ and $\dfrac{\partial y}{\partial t}$.

$$\frac{\partial x}{\partial t} = e^u \cos(t) = y \text{ and } \frac{\partial y}{\partial t} = -e^u \sin(t) = -x.$$

2. Calculate $\dfrac{\partial z}{\partial t}$ and express it purely in terms of x, y, $\dfrac{\partial z}{\partial x}$, and $\dfrac{\partial z}{\partial y}$.

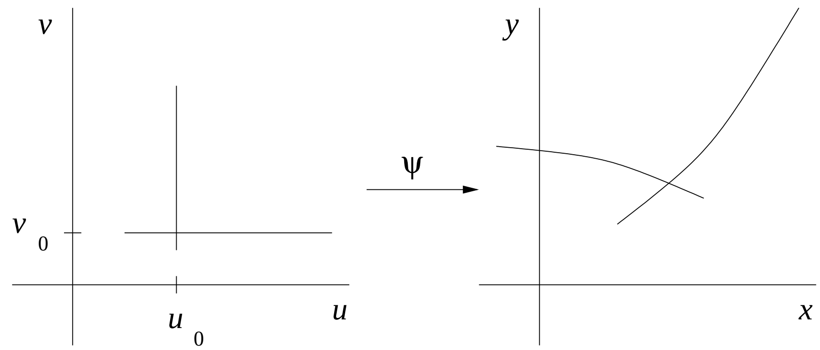
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x} y - \frac{\partial z}{\partial y} x.$$

3. Calculate $\dfrac{\partial}{\partial t}\left(\dfrac{\partial z}{\partial x} x\right)$ and express it purely in terms of x and y and partial derivatives of z.

Applying the Chain Rule, we have

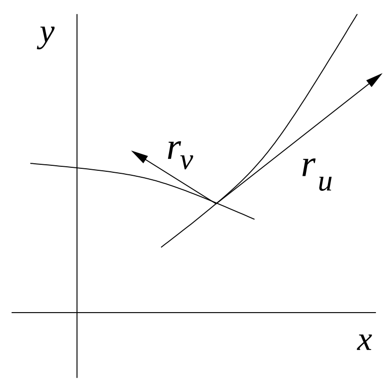
$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}x\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}x\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}x\right)\frac{\partial y}{\partial t}$$
$$= \left(\frac{\partial^2 z}{\partial x^2}x + \frac{\partial z}{\partial x}\right)y + \left(\frac{\partial^2 z}{\partial x\,\partial y}x\right)(-x) = \frac{\partial^2 z}{\partial x^2}xy - \frac{\partial^2 z}{\partial x\,\partial y}x^2 + \frac{\partial z}{\partial x}y \ .$$

VIII. (5) The figure to the right shows a change-of-coordinate function ψ, of the form $\psi(u,v) = (x(u,v), y(u,v))$.



(a) In the xy-coordinate system, sketch a possibility for what the vectors $\vec{r}_u = \dfrac{\partial x}{\partial u}\,\vec{\imath} + \dfrac{\partial y}{\partial u}\,\vec{\jmath}$ and $\vec{r}_v = \dfrac{\partial x}{\partial v}\,\vec{\imath} + \dfrac{\partial y}{\partial v}\,\vec{\jmath}$ might look like.

(b) Calculate the length of their cross product, and use it to write the relation between $dx\,dy$ and $du\,dv$.



$$\|\,\vec{r}_u \times \vec{r}_v\,\| = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \left\|\; 0\;\vec{\imath} + 0\;\vec{\jmath} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\vec{k}\;\right\| = \left|\;\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\;\right|$$

$$dx\,dy = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|\,du\,dv$$

IX. Consider the change-of-coordinate function $x = 3u$, $y = 2v$.
(5)
(a) Calculate the Jacobian of this change of coordinates.

$$\begin{vmatrix} \frac{\partial}{\partial u}(3u) & \frac{\partial}{\partial u}(2v) \\ \frac{\partial}{\partial v}(3u) & \frac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

(b) Find the curve in the uv-plane that corresponds to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

$$\frac{(3u)^2}{9} + \frac{(2v)^2}{4} = 1$$
$$u^2 + v^2 = 1$$

(c) Use this change of coordinates to find the area inside the ellipse, by calculating an integral in the uv-plane. Letting E be the region inside the ellipse and D the unit disk in the uv-plane, we have

$$\text{area}(E) = \iint_E dx\,dy = \iint_D 6\,du\,dv = 6 \iint_D du\,dv = 6\,\text{area}(D) = 6\pi$$