Mathematics 2443-006H Name (please print) ______________________

Examination II

March 13, 2008

Instructions: Give brief, clear answers.

I. (6) Let a be a positive number and let T be the triangle in the xy-plane bounded by $x = 0$, $y = 0$, and $x + y = a$. Find the centroid $(\overline{x}, \overline{y})$ of T (i. e. the center of mass, assuming that $\rho = 1$) of T. You may take it as obvious that the centroid lies on the line $y = x$, so it is only necessary to calculate one of $\overline{x}$ or $\overline{y}$.

II. (6) Let E be the region in the first octant bounded by the surfaces $x^2 + y^2 = 1$, $z = 1$, and $y + z = 1$ (so $z = 1$ forms the top of the solid). Sketch the region, and supply limits of integration, in cylindrical coordinates, for the integral $\iiint_E f(x, y, z)\, dV$.

III. (6) Evalute $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.

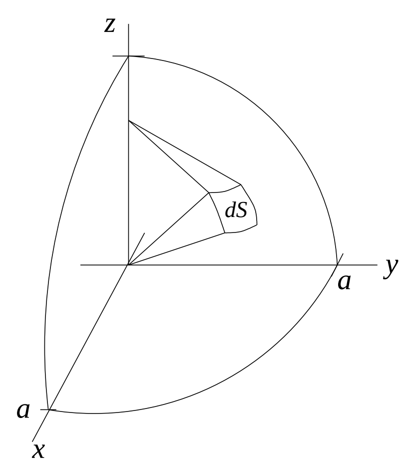
IV. (6) Find the surface area of the part of the paraboloid $y = 1 - x^2 - z^2$ that has $y \geq 0$.

V. (7) Find the z-coordinate $\overline{z}$ of the center of mass of the portion of the region E in the first octant that lies inside the sphere $x^2 + y^2 + z^2 = 4$, assuming that the density is proportional to the distance from the origin.

VI. (6) Let S be the sphere of radius a with center at the origin.

(a) The differential of surface area on S can be expressed in terms of $d\phi$ and $d\theta$. Using the picture shown to the right, explain why dS appears to be $a^2 \sin(\phi)\, d\phi\, d\theta$.

(b) Using the expression $dS = a^2 \sin(\phi)\, d\phi\, d\theta$, use a double integral in the variables ϕ and θ to calculate that the area of S is $4\pi a^2$.



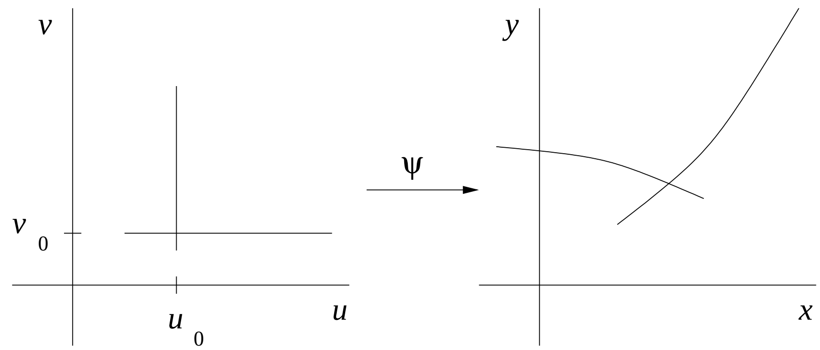
VII. (5) Let $x = e^u \sin(t)$, $y = e^u \cos(t)$, and $z = f(x, y)$.

1. Calculate $\dfrac{\partial x}{\partial t}$ and $\dfrac{\partial y}{\partial t}$.

2. Calculate $\dfrac{\partial z}{\partial t}$ and express it purely in terms of x, y, $\dfrac{\partial z}{\partial x}$, and $\dfrac{\partial z}{\partial y}$.

3. Calculate $\dfrac{\partial}{\partial t}\left(\dfrac{\partial z}{\partial x} x\right)$ and express it purely in terms of x and y and partial derivatives of z.

VIII. The figure to the right shows
(5) a change-of-coordinate function ψ, of the form $\psi(u,v) = (x(u,v), y(u,v))$.



(a) In the xy-coordinate system, sketch a possibility for what the vectors $\vec{r}_u = \dfrac{\partial x}{\partial u}\,\vec{\imath} + \dfrac{\partial y}{\partial u}\,\vec{\jmath}$ and $\vec{r}_v = \dfrac{\partial x}{\partial v}\,\vec{\imath} + \dfrac{\partial y}{\partial v}\,\vec{\jmath}$ might look like.

(b) Calculate the length of their cross product, and use it to write the relation between $dx\,dy$ and $du\,dv$.

IX. Consider the change-of-coordinate function $x = 3u$, $y = 2v$.
(5)

(a) Calculate the Jacobian of this change of coordinates.

(b) Find the curve in the uv-plane that corresponds to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

(c) Use this change of coordinates to find the area inside the ellipse, by calculating an integral in the uv-plane.