Mathematics 2443-006H Name (please print) ____________________

Final Examination

May 7, 2008

Instructions: Give brief, clear answers. Use theorems whenever possible.

I. (6) Verify the Divergence Theorem for the vector field $\vec{F}(x,y,z) = z\vec{k}$ on the sphere $x^2 + y^2 + z^2 = 1$ with the outward normal.

Using the formulas $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dD$ and $\vec{r}_\phi \times \vec{r}_\theta = a \sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$, we have

$$\iint_S z\vec{k} \cdot d\vec{S} = \iint_R z\vec{k} \cdot \sin(\phi)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})\, dA = \iint_R \sin(\phi) z^2\, dA$$
$$= \int_0^{2\pi}\int_0^{\pi} \sin(\phi)\cos^2(\phi)\, d\phi\, d\theta = \int_0^{2\pi} -\frac{1}{3}\cos^3(\phi)\Big|_0^{\pi}\, d\theta = \int_0^{2\pi} \frac{2}{3}\, d\theta = \frac{4}{3}\pi\ ,$$

while for the solid unit ball E,

$$\iiint_E \operatorname{div}(z\vec{k})\, dV = \iiint_E dV = \operatorname{vol}(E) = \frac{4}{3}\pi\ .$$

II. (6) Use Stokes' Theorem to calculate $\int_C (e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) \cdot d\vec{r}$, where C is the boundary of the portion of the surface $x + y + z = 1$ that lies in the first octant.

One calculates that $\operatorname{curl}(e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) = e^x\vec{k}$. Then, letting S be the portion of the plane in the first octant and using Stokes' Theorem, we have

$$\int_C (e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) \cdot d\vec{r} = \iint_S \operatorname{curl}(e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) \cdot d\vec{S} = \iint_S e^x\vec{k} \cdot d\vec{S}\ .$$

Now S is the graph of the function $z = 1-x-y$ over the domain R in the xy-plane given by $0 \le x \le 1$, $0 \le y \le 1 - x$. Using the formula for a surface integral on a surface that is the graph of a function, we find

$$\iint_S e^x\vec{k} \cdot d\vec{S} = \iint_R e^x\, dR = \int_0^1 \int_0^{1-y} e^x\, dx\, dy = \int_0^1 e^{1-y} - 1\, dy = (-e^{1-1} + e^{1-0}) - 1 = e - 2\ .$$

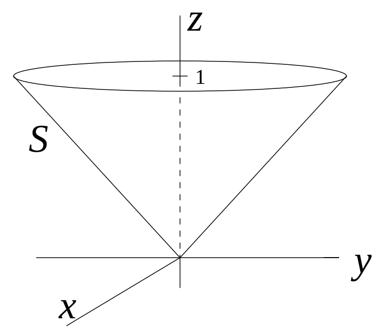
III. (6) Let f be a scalar function of three variables and let $\vec{F}$ be a vector field on a 3-dimensional domain. Writing $\vec{F}$ as $P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$, verify that $\operatorname{div}(f\vec{F}) = f\operatorname{div}(\vec{F}) + \nabla f \cdot \vec{F}$.

$$\operatorname{div}(f\vec{F}) = \operatorname{div}(fP\vec{\imath} + fQ\vec{\jmath} + fR\vec{k}) = \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$$
$$= \frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} + \frac{\partial f}{\partial y}Q + f\frac{\partial Q}{\partial y} + \frac{\partial f}{\partial z}R + f\frac{\partial R}{\partial z} = f\Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\Big) + \Big(\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R\Big)$$
$$= f\operatorname{div}(\vec{F}) + \Big(\frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}\Big) \cdot (P\vec{\imath} + Q\vec{\jmath} + R\vec{k}) = f\operatorname{div}(\vec{F}) + \nabla f \cdot \vec{F}\ .$$

IV. (6) The picture to the right shows a parameterization of the cone $z = \sqrt{x^2+y^2}$. It is parameterized by letting θ be the polar angle in the xy-plane, and h be the z-coordinate. The parameterization is

$$\begin{aligned} x &= h\cos(\theta) \\ y &= h\sin(\theta) \\ z &= h, \end{aligned}$$

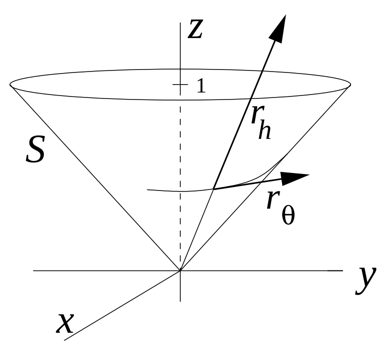
where the parameter domain R in the θh-plane consists of $0 \le \theta \le 2\pi$, $0 \le h \le 1$.



1. Calculate $\vec{r}_h$ and $\vec{r}_\theta$.

   $\vec{r}_h = \cos(\theta)\vec{\imath} + \sin(\theta)\vec{\jmath} + \vec{k}$ and $\vec{r}_\theta = -h\sin(\theta)\vec{\imath} + h\cos(\theta)\vec{\jmath}$.

2. For the line $0 \le h \le 1$, $\theta = \dfrac{\pi}{4}$ in R, draw the corresponding points on S. Do the same for the line $h = \dfrac{1}{2}$, $0 \le \theta \le \dfrac{\pi}{2}$. At the intersection point of these two curves on S, draw the vectors $\vec{r}_\theta$ and $\vec{r}_h$.



3. Is the upward normal (i. e. the one with positive $\vec{k}$-component) equal to $\vec{r}_\theta \times \vec{r}_h$ or to $\vec{r}_h \times \vec{r}_\theta$?

   It is $\vec{r}_h \times \vec{r}_\theta$.

V. (9) Let S be the surface given by the parametric equations $x = u^2$, $y = u\sin(v)$, $z = u\cos(v)$, with the parameter domain R given by $0 \le u \le 3$, $0 \le v \le \pi/2$.

1. Calculate dS in terms of dR.

   We have $\vec{r}_u = 2u\vec{\imath} + \sin(v)\vec{\jmath} + \cos(v)\vec{k}$ and $\vec{r}_v = u\cos(v)\vec{\jmath} - u\sin(v)\vec{k}$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2u & \sin(v) & \cos(v) \\ 0 & u\cos(v) & -u\sin(v) \end{vmatrix} = -u\vec{\imath} + 2u^2\sin(v)\vec{\jmath} + 2u^2\cos(v)\vec{k} \ ,$$

   so

$$dS = \|\vec{r}_u \times \vec{r}_v\|\, dR = \sqrt{u^2 + 4u^4\sin^2(v) + 4u^4\cos^2(v)}\, dR = \sqrt{u^2 + 4u^4}\, dR \ .$$

2. Find a normal vector to S at the point $(4, 1, \sqrt{3})$.

   At this point $u^2 = 4$ so u is 2 (because $0 \le u \le 3$). From the second coordinate, $1 = 2\sin(v)$, so $\sin(v) = \dfrac{1}{2}$ and therefore $v = \dfrac{\pi}{6}$. Evaluating $\vec{r}_u \times \vec{r}_v = -u\vec{\imath} + 2u^2\sin(v)\vec{\jmath} + 2u^2\cos(v)\vec{k}$ at $(2, \pi/6)$, we have the normal vector $-2\vec{\imath} + 4\vec{\jmath} + 4\sqrt{3}\vec{k}$.

3. Use the parameterization to calculate $\displaystyle\iint_S (x\vec{\imath} + z\vec{\jmath}) \cdot d\vec{S}$.

   Using the formula $\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dD$, we have

$$\iint_S (x\vec{\imath} + z\vec{\jmath}) \cdot d\vec{S} = \iint_R (x\vec{\imath} + z\vec{\jmath}) \cdot (-u\vec{\imath} + 2u^2\sin(v)\vec{\jmath} + 2u^2\cos(v)\vec{k})\, dR$$
$$= \iint_R -xu + z2u^2\cos(v)\, dR = \int_0^3 \int_0^{\pi/2} -u^3 + 2u^3\sin(v)\cos(v)\, dv\, du$$
$$= \int_0^3 -u^3\frac{\pi}{2} + \left(2u^3 \frac{\sin^2(v)}{2}\Big|_0^{\pi/2}\right) du = \int_0^3 \left(2 - \frac{\pi}{2}\right) u^3\, du = \frac{3^4}{4}\left(2 - \frac{\pi}{2}\right) \ .$$

VI. (6) Apply the Divergence Theorem to show that if S is the boundary of the solid E, then:

(a) The volume of E is $\displaystyle\frac{1}{3}\iint_S (x\vec{\imath} + y\vec{\jmath} + z\vec{k}) \cdot d\vec{S}$.

$$\frac{1}{3}\iint_S (x\vec{\imath} + y\vec{\jmath} + z\vec{k}) \cdot d\vec{S} = \frac{1}{3}\iiint_E \operatorname{div}(x\vec{\imath} + y\vec{\jmath} + z\vec{k})\, dV = \frac{1}{3}\iiint_E 3\, dV = \iiint_E dV = \operatorname{vol}(E) \ .$$

(b) If $\vec{n}$ is the unit outward normal on S, then $\displaystyle\iint_S D_{\vec{n}} f\ dS = \iiint_E \Delta f\, dV$, where $\vec{n}$ is the unit normal to the surfaces and Δf is the Laplacian $f_{xx} + f_{yy} + f_{zz}$.

The Divergence Theorem gives $\displaystyle\iint_S D_{\vec{n}} f\ dS = \iint_S \nabla f \cdot \vec{n}\ dS = \iint_S \nabla f \cdot d\vec{S} = \iiint_E \operatorname{div}(\nabla f)\, dV$. Since $\operatorname{div}(\nabla f) = \operatorname{div}(f_x\vec{\imath} + f_y\vec{\jmath} + f_z\vec{k}) = f_{xx} + f_{yy} + f_{zz} = \Delta f$, the last integral equals $\displaystyle\iiint_E \Delta f\, dV$.

VII. Five positive numbers x, y, z, u, and v are multiplied together. That is, the product function
(9) is $P = x\,y\,z\,u\,v$.

1. Assume that each of x, y, and z is increasing at 0.2 units per second, and each of u and v is decreasing at 0.1 units per second. Find the rate of change of the product at a moment when all of the numbers except v equal 1, and $v = 2$.

   The Chain Rule gives

$$\begin{aligned}\frac{d(xyzuv)}{dt} &= \frac{\partial}{\partial x}(xyzuv)\frac{dx}{dt} + \frac{\partial}{\partial y}(xyzuv)\frac{dy}{dt} + \frac{\partial}{\partial z}(xyzuv)\frac{dz}{dt} + \frac{\partial}{\partial u}(xyzuv)\frac{du}{dt} + \frac{\partial}{\partial v}(xyzuv)\frac{dv}{dt}\\ &= yzuv\frac{dx}{dt} + xzuv\frac{dy}{dt} + xyuv\frac{dz}{dt} + xyzv\frac{du}{dt} + xyzu\frac{dv}{dt}\end{aligned}$$

   Specializing to the moment with the given information, we obtain the rate of change

$$2 \cdot 0.2 + 2 \cdot 0.2 + 2 \cdot 0.2 - 2 \cdot 0.1 - 0.1 = 0.9\ .$$

2. Calculate the differential of P.

$$d(xyzuv) = yzuv\,dx + xzuv\,dy + xyuv\,dz + xyzv\,du + xyzu\,dv$$

3. This time, assume that each of x, y, and z is less than or equal to 1, and each of u and v is less than or equal to 5. Use the differential dP to estimate the maximum possible error in the computed product P that might result from rounding each number off to the nearest whole number.

   Rounding off to the nearest integer allows any of dx, etc., to be as large as 0.5. In each of the four-term products, x, y, and z are at most 1 and u and v are at most 5, so the maximum error is

$$25 \cdot 0.5 + 25 \cdot 0.5 + 25 \cdot 0.5 + 5 \cdot 0.5 + 5 \cdot 0.5 = 85/2\ .$$

VIII. For the function $\sin(2x + y + z)$, calculate each of the following.
(6)

(a) The directional derivative $(2, 1, 2)$ in the direction toward the origin.

   We have $\nabla(\sin(2x+y+z)) = 2\cos(2x+y+z)\vec{\imath} + \cos(2x+y+z)\vec{\jmath} + \cos(2x+y+z)\vec{k}$, so $\nabla(\sin(2x+y+z))(2,1,2) = 2\cos(7)\vec{\imath} + \cos(7)\vec{\jmath} + \cos(7)\vec{k}$. The vector from $(2, 1, 2)$ to the origin is $-2\vec{\imath} - jj - 2\vec{k}$, which has length 3, so a unit vector in that direction is $-\frac{2}{3}\vec{\imath} - \frac{1}{3}\vec{\jmath} - \frac{2}{3}\vec{k}$. To find the directional derivative, we calculate

$$\nabla(\sin(2x + y + z))(2, 1, 2) \cdot (-\frac{2}{3}\vec{\imath} - \frac{1}{3}\vec{\jmath} - \frac{2}{3}\vec{k}) = -\frac{7}{3}\cos(7)\ .$$

(b) The maximum rate of change at $(2, 1, 2)$, and the direction in which it occurs.

   The maximum rate of change is $\|\,\nabla(\sin(2x + y + z))(2, 1, 2)\,\| = \sqrt{6}\cos(7)$, and occurs in the direction of $\nabla(\sin(2x + y + z))(2, 1, 2) = 2\cos(7)\vec{\imath} + \cos(7)\vec{\jmath} + \cos(7)\vec{k}$.

IX. (6) For each of the following multiple integrals, rewrite the integral to change the order of integration as requested.

(a) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{0} f(x,y)\,dy\,dx$, write to integrate first with respect to x.

The domain of integration is the bottom half of the disk of radius $\sqrt{3}$, so changing the order of integration gives $\displaystyle\int_{-\sqrt{3}}^{0}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} f(x,y)\,dx\,dy$.

(b) $\displaystyle\int_0^1\int_0^1\int_0^y f(x,y,z)\,dz\,dx\,dy$, write to integrate first with respect to y, then with respect to x, then with respect to z.

The region of integration is the space above the square $[0,1]\times[0,1]$ in the xy-plane and below the plane $z=y$. For fixed z and x, the range of y is $z\le y\le 1$, and the possible xz-values are the square $[0,1]\times[0,1]$. So we have

$$\int_0^1\int_0^1\int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^1\int_z^1 f(x,y,z)\,dy\,dx\,dz\ .$$

X. (6) Let $P(x,y)$ and $Q(x,y)$ be functions with continuous partial derivatives, defined on the rectangle $[a,b]\times[c,d]$ in the xy-plane. Verify the following facts. (Do not try to use Green's Theorem; it is not the best way to verify these facts, and that approach would not be appropriate anyway, since these facts are some of the key steps in the proof of Green's Theorem).

1. $\displaystyle\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d Q(b,y)\,dy - \int_c^d Q(a,y)\,dy$

$$\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d\int_a^b \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d \frac{\partial Q}{\partial x}\Big|_{x=a}^{x=b}\,dy$$
$$= \int_c^d Q(b,y) - Q(a,y)\,dy = \int_c^d Q(b,y)\,dy - \int_c^d Q(a,y)\,dy$$

where we used the Fundamental Theorem of Calculus to integrate $\dfrac{\partial Q}{\partial x}$.

2. $\displaystyle\int_C P\,dx + Q\,dy = \int_c^d Q(b,y)\,dy$, where C is the straight line segment from (b,c) to (b,d).

Parameterizing C as $x=b$, $y=t$ for $c\le t\le d$, we have $dx = 0\,dt$ and $dy=dt$, so

$$\int_C P\,dx + Q\,dy = \int_c^d P(b,t)\cdot 0\,dt + Q(b,t)\,dt = \int_c^d Q(b,t)\,dt = \int_c^d Q(b,y)\,dy\ .$$

XI. (6) Calculate $\int_T (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r}$, where T is the equilateral triangle that has one side the straight line from $(1, 1)$ to $(201, 1)$ and lies in quadrants I and IV.

Using Green's Theorem, $\int_C (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r} = \iint_T \frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y}\, dA = \iint_T -2\, dA$, which is -2 times the area of T. Since T has base of length 200, a little geometry shows that the area of T is $(\sqrt{3}/2)(200)^2/2 = 10,000\sqrt{3}$, so the answer is $-20,000\sqrt{3}$.

XII. (6) A certain conservative vector field $\vec{F}$ is of the form $(y^3 + 1)^{20} \cos(x)\vec{\imath} + g(x, y)\vec{\jmath}$, for some function $g(x, y)$. Let C be the upper half of the unit circle, oriented clockwise. Calculate $\int_C \vec{F} \cdot d\vec{r}$.

The vector field is defined on the entire plane, and since it is conservative, the line integral is path-independent. So instead of using C, we may use the straight path C_1 which goes from $(-1, 0)$ to $(1, 0)$ along the x-axis. That is,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} ((y^3 + 1)^{20} \cos(x)\vec{\imath} + g(x, y)\vec{\jmath}) \cdot d\vec{r} = \int_{C_1} \cos(x)\vec{\imath} + g(x, y)\vec{\jmath}) \cdot d\vec{r} \ ,$$

since $y = 0$ on C_1. We also have

$$\int_{C_1} \cos(x)\vec{\imath} + g(x, y)\vec{\jmath}) \cdot d\vec{r} = \int_{C_1} \cos(x)\, dx + g(x, y)\, dy = \int_{C_1} \cos(x)\, dx \ ,$$

since on C_1, $dy = 0\, dt$ for any parameterization. Taking the parameterization $x = t$, we have

$$\int_{C_1} \cos(x)\, dx = \int_{-1}^{1} \cos(t)\, dt = 2 \sin(1) \ .$$