Mathematics 2443-006H Name (please print) ______________________

Final Examination

May 7, 2008

Instructions: Give brief, clear answers. Use theorems whenever possible.

I. (6) Verify the Divergence Theorem for the vector field $\vec{F}(x, y, z) = z\vec{k}$ on the sphere $x^2 + y^2 + z^2 = 1$ with the outward normal.

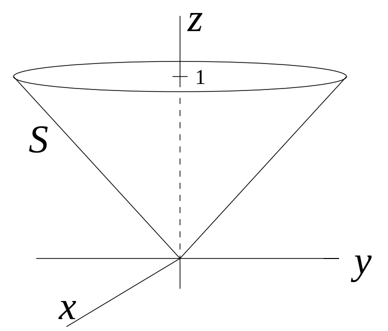
II. (6) Use Stokes' Theorem to calculate $\int_C (e^{-x}\vec{\imath} + e^x\vec{\jmath} + e^z\vec{k}) \cdot d\vec{r}$, where C is the boundary of the portion of the surface $x + y + z = 1$ that lies in the first octant.

III. (6) Let f be a scalar function of three variables and let $\vec{F}$ be a vector field on a 3-dimensional domain. Writing $\vec{F}$ as $P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$, verify that $\operatorname{div}(f\vec{F}) = f \operatorname{div}(\vec{F}) + \nabla f \cdot \vec{F}$.

IV. (6) The picture to the right shows a parameterization of the cone $z = \sqrt{x^2 + y^2}$. It is parameterized by letting θ be the polar angle in the xy-plane, and h be the z-coordinate. The parameterization is

$$\begin{aligned} x &= h\cos(\theta) \\ y &= h\sin(\theta) \\ z &= h, \end{aligned}$$

where the parameter domain R in the θh-plane consists of $0 \le \theta \le 2\pi$, $0 \le h \le 1$.



1. Calculate $\vec{r}_h$ and $\vec{r}_\theta$.
2. For the line $0 \le h \le 1$, $\theta = \dfrac{\pi}{4}$ in R, draw the corresponding points on S. Do the same for the line $h = \dfrac{1}{2}$, $0 \le \theta \le \dfrac{\pi}{2}$. At the intersection point of these two curves on S, draw the vectors $\vec{r}_\theta$ and $\vec{r}_h$.
3. Is the upward normal (i. e. the one with positive $\vec{k}$-component) equal to $\vec{r}_\theta \times \vec{r}_h$ or to $\vec{r}_h \times \vec{r}_\theta$?

V. (9) Let S be the surface given by the parametric equations $x = u^2$, $y = u\sin(v)$, $z = u\cos(v)$, with the parameter domain R given by $0 \le u \le 3$, $0 \le v \le \pi/2$.

1. Calculate dS in terms of dR.
2. Find a normal vector to S at the point $(4, 1, \sqrt{3})$.
3. Use the parameterization to calculate $\iint_S (x\vec{\imath} + z\vec{\jmath}) \cdot d\vec{S}$.

VI. (6) Apply the Divergence Theorem to show that if S is the boundary of the solid E, then:

(a) The volume of E is $\dfrac{1}{3}\iint_S (x\vec{\imath} + y\vec{\jmath} + z\vec{k}) \cdot d\vec{S}$.

(b) If $\vec{n}$ is the unit outward normal on S, then $\iint_S D_{\vec{n}} f \, dS = \iiint_E \Delta f \, dV$, where $\vec{n}$ is the unit normal to the surfaces and Δf is the Laplacian $f_{xx} + f_{yy} + f_{zz}$.

VII. (9) Five positive numbers x, y, z, u, and v are multiplied together. That is, the product function is $P = x\,y\,z\,u\,v$.

1. Assume that each of x, y, and z is increasing at 0.2 units per second, and each of u and v is decreasing at 0.1 units per second. Find the rate of change of the product at a moment when all of the numbers except v equal 1, and $v = 2$.
2. Calculate the differential of P.
3. This time, assume that each of x, y, and z is less than or equal to 1, and each of u and v is less than or equal to 5. Use the differential dP to estimate the maximum possible error in the computed product P that might result from rounding each number off to the nearest whole number.

VIII. (6) For the function $\sin(2x + y + z)$, calculate each of the following.

(a) The directional derivative $(2, 1, 2)$ in the direction toward the origin.

(b) The maximum rate of change at $(2, 1, 2)$, and the direction in which it occurs.

IX. (6) For each of the following multiple integrals, rewrite the integral to change the order of integration as requested.

(a) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{0} f(x, y)\, dy\, dx$, write to integrate first with respect to x.

(b) $\displaystyle\int_0^1 \int_0^1 \int_0^y f(x, y, z)\, dz\, dx\, dy$, write to integrate first with respect to y, then with respect to x, then with respect to z.

X. (6) Let $P(x, y)$ and $Q(x, y)$ be functions with continuous partial derivatives, defined on the rectangle $[a, b]\times[c, d]$ in the xy-plane. Verify the following facts. (Do not try to use Green's Theorem; it is not the best way to verify these facts, and that approach would not be appropriate anyway, since these facts are some of the key steps in the proof of Green's Theorem).

1. $\displaystyle\iint_D \frac{\partial Q}{\partial x}\, dA = \int_c^d Q(b, y)\, dy - \int_c^d Q(a, y)\, dy$
2. $\displaystyle\int_C P\, dx + Q\, dy = \int_c^d Q(b, y)\, dy$, where C is the straight line segment from (b, c) to (b, d).

XI. (6) Calculate $\displaystyle\int_T (y\vec{\imath} - x\vec{\jmath}) \cdot d\vec{r}$, where T is the equilateral triangle that has one side the straight line from $(1, 1)$ to $(201, 1)$ and lies in quadrants I and IV.

XII. (6) A certain conservative vector field $\vec{F}$ is of the form $(y^3 + 1)^{20}\cos(x)\vec{\imath} + g(x, y)\vec{\jmath}$, for some function $g(x, y)$. Let C be the upper half of the unit circle, oriented clockwise. Calculate $\int_C \vec{F} \cdot d\vec{r}$.