Mathematics 2423-001H Name (please print) ______________________

Examination III

April 26, 2007

I. Evaluate the following integrals:
(20)

1. $\int \sin^3(3x)\ \cos^2(3x)\,dx$

   $\int \sin^3(3x)\ \cos^2(3x)\,dx = \int \sin(3x)\,(1-\cos^2(3x))\cos^2(3x)\,dx = \int \sin(3x)\cos^2(3x)-\sin(3x)\cos^4(3x)\,dx = -\frac{\cos^3(3x)}{9}+\frac{\cos^5(3x)}{15}+C$

2. $\int \sin^2(x)\ \cos^2(x)\,dx$

   $\int \sin^2(x)\ \cos^2(x)\,dx = \int \frac{1}{2}(1-\cos(2x))\,\frac{1}{2}(1+\cos(2x))\,dx = \int \frac{1}{4}(1-\cos^2(2x))\,dx = \int \frac{1}{4}-\frac{1}{4}\frac{1}{2}(1+\cos(4x))\,dx = \int \frac{1}{8}-\frac{1}{8}\cos(4x)\,dx = \frac{x}{8}-\frac{1}{32}\sin(4x)+C$

3. $\int \sin^{-1}(x)\,dx.$

   Using integration by parts with $u=\sin^{-1}(x)$, $du=\frac{dx}{\sqrt{1-x^2}}$, $v=x$, $dv=dx$, we have $\int \sin^{-1}(x)\,dx = x\ \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\ \sin^{-1}(x) + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = x\ \sin^{-1}(x)+\sqrt{1-x^2}+C$

4. $\int \frac{x^3}{x^2+1}\,dx.$

   Dividing gives $\frac{x^3}{x^2+1} = x-\frac{x}{x^2+1}$, so we have $\int \frac{x^3}{x^2+1}\,dx = \int x-\frac{x}{x^2+1}\,dx = \frac{x^2}{2}-\frac{1}{2}\ln(x^2+1)+C$. (This can also be solved using the trigonometric substitution $x=\tan(\theta)$, but that method seems a bit more complicated in this example.)

5. $\int \sinh(\sqrt{x})\,dx.$

   First we substitute $w^2=x$ and $2w\,dw=dx$, obtaining $\int 2w\ \sinh(w)\,dw$. Then, using integration by parts with $u=2w$, $du=2\,dw$, $v=\cosh(w)$, $dv=\sinh(w)\,dw$, we have $\int 2w\ \sinh(w)\,dw = 2w\cosh(w)-\int 2\cosh(w)\,dw = 2w\cosh(w)-2\sinh(w)+C = 2\sqrt{x}\cosh(\sqrt{x})-2\sinh(\sqrt{x})+C$

II. (6) Calculate the following derivatives:

1. $\dfrac{d}{dx}(5^{-1/x})$

$$\frac{d}{dx}(5^{-1/x}) = \frac{d}{dx}(e^{-\ln(5)/x}) = e^{-\ln(5)/x} \cdot \ln(5)/x^2 = 5^{-1/x}\ln(5)/x^2.$$

2. $\dfrac{d}{dx}(\log_3(x^2-4))$

$$\frac{d}{dx}(\log_3(x^2-4)) = \frac{d}{dx}\Big(\frac{1}{\ln(3)}\ln(x^2-4)\Big) = \frac{1}{\ln(3)}\,\frac{1}{x^2-4}\cdot 2x = \frac{2}{\ln(3)}\,\frac{x}{x^2-4}$$

III. (6) Use the method of partial fractions to evaluate $\displaystyle\int \frac{1}{x^4+x^2}\,dx$.

Setting up partial fractions, we have

$$\frac{1}{x^4+x^2} = \frac{1}{(x^2+1)x^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x} + \frac{D}{x^2}$$
$$1 = (Ax+B)x^2 + Cx(x^2+1) + D(x^2+1)$$

Putting $x = 0$ gives $D = 1$. So we have

$$1 = (Ax+B)x^2 + Cx(x^2+1) + x^2 + 1$$
$$-x^2 = (Ax+B)x^2 + Cx(x^2+1) = (A+C)x^3 + Bx^2 + Cx$$

So we have $C = 0$, $B = -1$, $0 = A + C = A + 0 = A$. Therefore

$$\int \frac{1}{x^4+x^2}\,dx = \int \frac{-1}{x^2+1} + \frac{1}{x^2}\,dx = -\tan^{-1}(x) - \frac{1}{x} + C$$

(Note: there is also a clever way to find the partial fractions quickly:

$$\frac{1}{x^4+x^2} = \frac{1}{(x^2+1)x^2} = \frac{x^2+1}{(x^2+1)x^2} - \frac{x^2}{(x^2+1)x^2} = \frac{1}{x^2} - \frac{1}{x^2+1}\ )$$

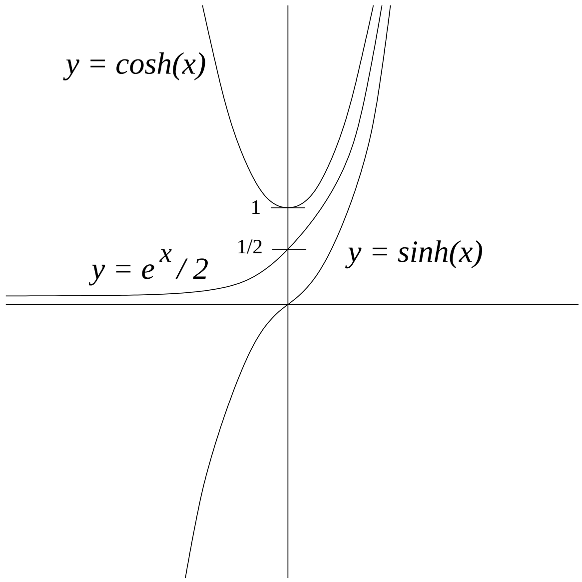
IV. (6) Use the substitution $x = \tan(\theta)$ to evaluate $\displaystyle\int \frac{1}{x^4+x^2}\,dx$. Make it clear how you are obtaining the answer in terms of x.

Putting $x = \tan(\theta)$ and $dx = \sec^2(\theta)\,d\theta$, we have

$$\int \frac{1}{x^4+x^2}\,dx = \int \frac{\sec^2(\theta)}{\tan^4(\theta)+\tan^2(\theta)}\,dx = \int \frac{\sec^2(\theta)}{\tan^2(\theta)(\tan^2(\theta)+1)}\,dx$$
$$= \int \frac{\sec^2(\theta)}{\tan^2(\theta)\sec^2(\theta)}\,dx = \int \cot^2(\theta)\,dx = \int \csc^2(\theta) - 1\,dx = -\cot(\theta) - \theta\ + C.$$

But $\cot(\theta) = 1/\tan(\theta) = 1/x$, so the last expression equals $-\dfrac{1}{x} - \tan^{-1}(x) + C$.

V. (4) On a single coordinate system, sketch the graphs of $y = \sinh(x)$, $y = \cosh(x)$, and $y = e^x/2$. Make the relation between them clear.



VI. (6) Evaluate the following limits. Show your reasoning.

1. $\lim_{x\to\infty} x^{\ln(2)/(1+\ln(x))}$

$\lim_{x\to\infty} x^{\ln(2)/(1+\ln(x))} = \lim_{x\to\infty} e^{\ln(x^{\ln(2)/(1+\ln(x))})} = \lim_{x\to\infty} e^{\frac{\ln(2)\ln(x)}{1+\ln(x)}}$. Applying l'Hôpital's rule gives $\lim_{x\to\infty} e^{\frac{\ln(2)\ln(x)}{1+\ln(x)}} = \lim_{x\to\infty} e^{\frac{\ln(2)/x}{1/x}} = \lim_{x\to\infty} e^{\ln(2)} = \lim_{x\to\infty} 2 = 2.$

2. $\lim_{h\to 0} \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

Applying l'Hôpital's rule twice, we have

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$
$$= \lim_{h\to 0} \frac{f''(x+h) + f''(x-h)}{2} = \frac{f''(x) + f''(x)}{2} = f''(x) \ .$$

VII. (8) Suppose that $f(x)$ is a function whose second derivative $f''(x)$ exists and is continuous. Define $E(h)$ by the formula $f(a+h) = f(a) + f'(a)h + E(h)$. That is, $E(h)$ is the error of linear approximation.

1. Use integration by parts to calculate that $E(h) = \displaystyle\int_0^h (h-t)\, f''(a+t)\, dt$.

   Using $u = h - t$, $du = -dt$, $dv = f''(a+t)\, dt$, $v = f'(a+t)$ gives

$$\int_0^h (h-t)\, f''(a+t)\, dt = (h-t)f'(a+t)\Big|_0^h + \int_0^h f'(a+t)\, dt = -f'(a)h + f(a+h) - f(a) = E(h)\ .$$

2. Let m be the minimum and M the maximum of f'' on the interval $[a, a+h]$, so that

$$m \le f''(a+t) \le M$$

for $0 \le t \le h$. Notice that

$$(h-t)\, m \le (h-t)\, f''(a+t) \le (h-t)\, M$$

for $0 \le t \le h$. Now, show that $\frac{1}{2}h^2\, m \le E(h) \le \frac{1}{2}h^2\, M$.

   Since integration preserves inequalities, we have

$$\int_0^h (h-t)\, m\, dt \le \int_0^h (h-t)\, f''(a+t)\, dt \le \int_0^h (h-t)\, M\, dt\ .$$

   Now $\displaystyle\int_0^h (h-t)m\, dt = -\frac{(h-t)^2}{2}m\Big|_0^h = \frac{mh^2}{2}$, and similarly $\displaystyle\int_0^h (h-t)M\, dt = \frac{Mh^2}{2}$. So the last inequalities become

$$\frac{mh^2}{2} \le E(h) \le \frac{Mh^2}{2}$$
$$m \le \frac{2}{h^2}E(h) \le M$$

3. Use the Intermediate Value Theorem to show that there exists c in $[a, a+h]$ so that $E(h) = \frac{1}{2}f''(c)\, h$.

   Since $\frac{2}{h^2}E(h)$ lies between the function values m and M of $f''(a+t)$, where t lies in the interval $[0, h]$, the Intermediate Value Theorem shows that there exists some number c between 0 and h so that $f''(c) = \frac{2}{h^2}E(h)$. That is, $E(h) = \frac{1}{2}f''(c)\, h^2$.