I. Evaluate the following integrals:

(20)
1. \( \int \sin^3(3x) \cos^2(3x) \, dx \)

\[
\int \sin^3(3x) \cos^2(3x) \, dx = \int \sin(3x)(1-\cos^2(3x))\cos^2(3x) \, dx = \int \sin(3x)\cos^2(3x)-\sin(3x)\cos^4(3x) \, dx = -\frac{\cos^3(3x)}{9} + \frac{\cos^5(3x)}{15} + C
\]

2. \( \int \sin^2(x) \cos^2(x) \, dx \)

\[
\int \sin^2(x) \cos^2(x) \, dx = \int \frac{1}{2}(1-\cos(2x)) \frac{1}{2}(1+\cos(2x)) \, dx = \int \frac{1}{4}(1-\cos^2(2x)) \, dx = \int \frac{1}{4} - \frac{1}{4} \frac{1}{2}(1+\cos(4x)) \, dx = \int \frac{1}{8} - \frac{1}{8} \cos(4x) \, dx = \frac{x}{8} - \frac{1}{32} \sin(4x) + C
\]

3. \( \int \sin^{-1}(x) \, dx \).

Using integration by parts with \( u = \sin^{-1}(x) \), \( du = \frac{dx}{\sqrt{1-x^2}} \), \( v = x \), \( dv = dx \), we have \( \int \sin^{-1}(x) \, dx = x \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \sin^{-1}(x) + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \, dx = x \sin^{-1}(x) + \sqrt{1-x^2} + C \)

4. \( \int \frac{x^3}{x^2 + 1} \, dx \).

Dividing gives \( \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1} \), so we have \( \int \frac{x^3}{x^2 + 1} \, dx = \int x - \frac{x}{x^2 + 1} \, dx = \frac{x^2}{2} - \frac{1}{2} \ln(x^2+1) + C \).

(This can also be solved using the trigonometric substitution \( x = \tan(\theta) \), but that method seems a bit more complicated in this example.)

5. \( \int \sinh(\sqrt{x}) \, dx \).

First we substitute \( w^2 = x \) and \( 2w \, dw = dx \), obtaining \( \int 2w \sinh(w) \, dw \). Then, using integration by parts with \( u = 2w \), \( du = 2 \, dw \), \( v = \cosh(w) \), \( dv = \sinh(w) \, dw \), we have \( \int 2w \sinh(w) \, dw = 2w \cosh(w) - \int 2 \cosh(w) \, dw = 2w \cosh(w) - 2 \sinh(w) + C = 2\sqrt{x} \cosh(\sqrt{x}) - 2 \sinh(\sqrt{x}) + C \)
II. Calculate the following derivatives:

(6)

1. \( \frac{d}{dx} (5^{-1/x}) \)

\[
\frac{d}{dx} (5^{-1/x}) = \frac{d}{dx} (e^{-\ln(5)/x}) = e^{-\ln(5)/x} \cdot \frac{\ln(5)}{x^2} = 5^{-1/x} \ln(5)/x^2.
\]

2. \( \frac{d}{dx} (\log_3(x^2 - 4)) \)

\[
\frac{d}{dx} (\log_3(x^2 - 4)) = \frac{d}{dx} \left( \frac{1}{\ln(3)} \ln(x^2 - 4) \right) = \frac{1}{\ln(3)} \frac{1}{x^2 - 4} \cdot 2x = \frac{2}{\ln(3)} \frac{x}{x^2 - 4}
\]

III. Use the method of partial fractions to evaluate \( \int \frac{1}{x^4 + x^2} \, dx \).

(6)

Setting up partial fractions, we have

\[
\frac{1}{x^4 + x^2} = \frac{1}{(x^2 + 1)x^2} = \frac{A x + B}{x^2 + 1} + \frac{C}{x} + \frac{D}{x^2}
\]

\[1 = (Ax + B)x^2 + Cx(x^2 + 1) + D(x^2 + 1)\]

Putting \( x = 0 \) gives \( D = 1 \). So we have

\[-x^2 = (Ax + B)x^2 + Cx(x^2 + 1) = x(x^2 + 1) = (A + C)x^3 + Bx^2 + Cx\]

So we have \( C = 0, B = -1, 0 = A + C = A + 0 = A \). Therefore

\[
\int \frac{1}{x^4 + x^2} \, dx = \int \frac{-1}{x^2 + 1} + \frac{1}{x^2} \, dx = -\tan^{-1}(x) - \frac{1}{x} + C
\]

(Note: there is also a clever way to find the partial fractions quickly:

\[
\frac{1}{x^4 + x^2} = \frac{1}{(x^2 + 1)x^2} = \frac{x^2 + 1}{(x^2 + 1)x^2} - \frac{x^2}{(x^2 + 1)x^2} = \frac{1}{x^2} - \frac{1}{x^2 + 1}
\]

IV. Use the substitution \( x = \tan(\theta) \) to evaluate \( \int \frac{1}{x^4 + x^2} \, dx \). Make it clear how you are obtaining the answer in terms of \( x \).

(6)

Putting \( x = \tan(\theta) \) and \( dx = \sec^2(\theta) \, d\theta \), we have

\[
\int \frac{1}{x^4 + x^2} \, dx = \int \frac{\sec^2(\theta)}{\tan^4(\theta) + \tan^2(\theta)} \, d\theta = \int \frac{\sec^2(\theta)}{\tan^2(\theta)(\tan^2(\theta) + 1)} \, d\theta
\]

\[
= \int \frac{\sec^2(\theta)}{\tan^2(\theta) \sec^2(\theta)} \, d\theta = \int \cot^2(\theta) \, d\theta = \int \csc^2(\theta) - 1 \, d\theta = -\cot(\theta) - \theta + C.
\]

But \( \cot(\theta) = 1/\tan(\theta) = 1/x \), so the last expression equals \( -\frac{1}{x} - \tan^{-1}(x) + C \).
V. On a single coordinate system, sketch the graphs of \( y = \sinh(x) \), \( y = \cosh(x) \), and \( y = e^x/2 \). Make the relation between them clear.

![Graph of \( y = \sinh(x) \), \( y = \cosh(x) \), and \( y = e^x/2 \) on a single coordinate system.]

VI. Evaluate the following limits. Show your reasoning.

1. \[ \lim_{x \to \infty} x^{\ln(2)/(1 + \ln(x))} \]

   \[
   \lim_{x \to \infty} x^{\ln(2)/(1 + \ln(x))} = \lim_{x \to \infty} e^{\ln(x^{\ln(2)/(1 + \ln(x))})} = \lim_{x \to \infty} e^{\ln(2) / x}.
   \]

   Applying l'Hôpital's rule gives

   \[
   \lim_{x \to \infty} e^{\ln(2) / x} = \lim_{x \to \infty} e^{\ln(2)/x} = \lim_{x \to \infty} e^{\ln(2)} = \lim_{x \to \infty} 2 = 2.
   \]

2. \[ \lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} \]

   Applying l'Hôpital's rule twice, we have

   \[
   \lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = \lim_{h \to 0} \frac{f'(x + h) - f'(x - h)}{2h} = \frac{f''(x) + f''(x)}{2} = f''(x).
   \]
VII. Suppose that \( f(x) \) is a function whose second derivative \( f''(x) \) exists and is continuous. Define \( E(h) \) by the formula \( f(a + h) = f(a) + f'(a)h + E(h) \). That is, \( E(h) \) is the error of linear approximation.

1. Use integration by parts to calculate that \( E(h) = \int_0^h (h - t) f''(a + t) \, dt \).

Using \( u = h - t, \, du = -dt, \, dv = f''(a + t) \, dt, \, v = f'(a + t) \) gives

\[
\int_0^h (h - t) f''(a + t) \, dt = (h - t)f'(a + t) \bigg|_0^h + \int_0^h f'(a + t) \, dt = -f'(a)h + f(a + h) - f(a) = E(h) .
\]

2. Let \( m \) be the minimum and \( M \) the maximum of \( f'' \) on the interval \([a, a + h]\), so that

\[ m \leq f''(a + t) \leq M \]

for \( 0 \leq t \leq h \). Notice that

\[ (h - t) m \leq (h - t) f''(a + t) \leq (h - t) M \]

for \( 0 \leq t \leq h \). Now, show that \( \frac{1}{2}h^2 m \leq E(h) \leq \frac{1}{2}h^2 M \).

Since integration preserves inequalities, we have

\[
\int_0^h (h - t) m \, dt \leq \int_0^h (h - t) f''(a + t) \, dt \leq \int_0^h (h - t) M \, dt .
\]

Now \( \int_0^h (h - t)m \, dt = -\frac{(h - t)^2}{2}m \big|_0^h = \frac{mh^2}{2} \), and similarly \( \int_0^h (h - t)M \, dt = \frac{Mh^2}{2} \). So the last inequalities become

\[
\frac{mh^2}{2} \leq E(h) \leq \frac{Mh^2}{2} \]

\[ m \leq \frac{2}{h^2} E(h) \leq M \]

3. Use the Intermediate Value Theorem to show that there exists \( c \) in \([a, a + h]\) so that \( E(h) = \frac{1}{2}f''(c) h \).

Since \( \frac{2}{h^2}E(h) \) lies between the function values \( m \) and \( M \) of \( f''(a + t) \), where \( t \) lies in the interval \([0, h]\), the Intermediate Value Theorem shows that there exists some number \( c \) between 0 and \( h \) so that \( f''(c) = \frac{2}{h^2}E(h) \). That is, \( E(h) = \frac{1}{2}f''(c) h^2 \).