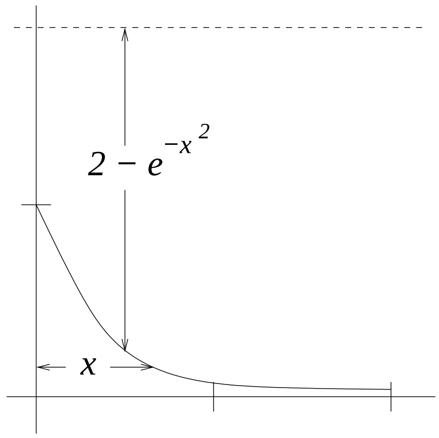
Mathematics 2423-001H Name (please print) ____________________

Examination II

March 29, 2007

Instructions: Give brief, clear answers. It is not expected that most people will be able to answer all the questions, just do what you can in 75 minutes.

I. (7) Let R be the region bounded by $y = e^{-x^2}$, $x = 0$, and $x = 2$.



1. Calculate the volume produced when this region is rotated about the y-axis.

   $V = \displaystyle\int_0^2 2\pi x\, e^{-x^2}\, dx$. Substituting $u = -x^2$, with $du = -2x\, dx$ so $2x\, dx = -du$, we have $V = \displaystyle\int_0^{-4} -\pi e^u\, du = -\pi e^u \Big|_0^{-4} = \pi(1 - e^{-4})$.

2. Write an integral whose value is the volume produced when R is rotated about the line $y = 2$, but do *not* evaluate it.

   The cross sections are annuli with outer radius 2 and inner radius $2 - e^{-x^2}$, so $V = \displaystyle\int_0^2 \pi\,(4 - (2 - e^{-x^2})^2)\, dx = \pi \int_0^2 4e^{-x^2} - e^{-2x^2}\, dx$.

II. (6) Calculate the following derivatives:

1. $\dfrac{d}{dx}(5^{-1/x})$

   $\dfrac{d}{dx}(5^{-1/x}) = \dfrac{d}{dx}(e^{-\ln(5)/x}) = e^{-\ln(5)/x} \cdot \ln(5)/x^2 = 5^{-1/x}\ln(5)/x^2$.

2. $\dfrac{d}{dx}(\log_3(x^2 - 4))$

   $\dfrac{d}{dx}(\log_3(x^2 - 4)) = \dfrac{d}{dx}\Big(\dfrac{1}{\ln(3)}\ln(x^2 - 4)\Big) = \dfrac{1}{\ln(3)}\,\dfrac{1}{x^2 - 4} \cdot 2x = \dfrac{2}{\ln(3)}\,\dfrac{x}{x^2 - 4}$

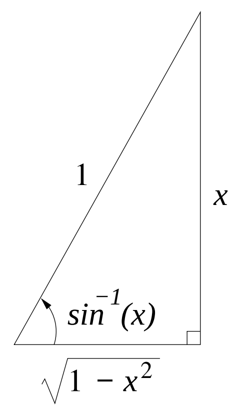
III. (5) Use the definition of $\ln(x)$ and the fact that integration is additive on unions of domains to verify that $\ln(ab) = \ln(a) + \ln(b)$.

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\, dt = \int_1^{a} \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt \ .$$

Substituting $u = t/a$ and $a\,du = dt$ in the second integral, we have

$$\int_1^{a} \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt = \int_1^{a} \frac{1}{t}\, dt + \int_1^{b} \frac{1}{au}\, a\,du = \int_1^{a} \frac{1}{t}\, dt + \int_1^{b} \frac{1}{u}\, du = \ln(a) + \ln(b) \ .$$

IV. (5) The following problem concerns the function $\sin^{-1}(x)$, which is the inverse function of the function $f(x)$ with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ given by $f(x) = \sin(x)$.



1. Draw a right triangle containing an angle of $\sin^{-1}(x)$, and use it to find $\cos(\sin^{-1}(x))$.

   $\cos(\sin^{-1}(x)) = \sqrt{1 - x^2}$

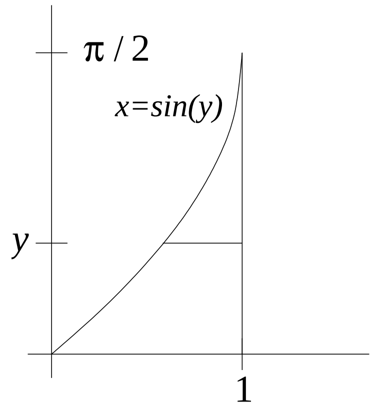
2. Differentiate the equation $\sin(\sin^{-1}(x)) = x$ and simplify to find the derivative of $\sin^{-1}(x)$.

$$\begin{aligned} \sin(\sin^{-1}(x)) &= x \\ \cos(\sin^{-1}(x)) \cdot \frac{d}{dx}\big(\sin^{-1}(x)\big) &= 1 \\ \frac{d}{dx}\big(\sin^{-1}(x)\big) &= \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

V. (3) Give a precise definition of what it means to say that a function f is *injective* (also called *one-to-one*).

To say that f is injective means that if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

VI. Calculate the area of the region bounded by $y = \sin^{-1}(x)$, $x = 1$, and $y = 0$.
(4)



The points on the graph of the equation $y = \sin^{-1}(x)$ satisfy $x = \sin(y)$, so the length of the horizontal segment in the figure above is $1 - \sin(y)$. So we can find the area by integrating with respect to y:

$$A = \int_0^{\pi/2} 1 - \sin(y)\, dy = y + \cos(y)\Big|_0^{\pi/2} = \pi/2 - 1\ .$$

VII. Find the domain of the function $\ln(e^x - 2)$.
(3)

We need $e^x - 2 > 0$ or $e^x > 2$. Since the logarithm function is increasing, this is the same as $\ln(e^x) > \ln(2)$, i. e. the domain is $x > \ln(2)$ or $(\ln(2), \infty)$.

VIII. Evalulate the following integrals.
(12)

1. $\displaystyle\int \frac{e^x + 1}{e^x}\, dx$

$$\int \frac{e^x + 1}{e^x}\, dx = \int 1 + \frac{1}{e^x}\, dx = \int 1 + e^{-x}\, dx = x - e^{-x} + C.$$

2. $\displaystyle\int \frac{e^x}{e^x + 1}\, dx$

Putting $u = e^x + 1$ and $du = e^x\, dx$, we have $\displaystyle\int \frac{e^x}{e^x + 1}\, dx = \int \frac{du}{u}\, dx = \ln(e^x + 1) + C.$

3. $\displaystyle\int \frac{t^2}{5 + t^6}\, dt$

First, factor out 5 from the denominator: $\displaystyle\int \frac{t^2}{5 + t^6}\, dt = \frac{1}{5}\int \frac{1}{1 + t^6/5}\, dt$. Now, putting $u = t^3/\sqrt{5}$ and $du = 3t^2/\sqrt{5}\, dt$, we have $\displaystyle\frac{1}{5}\int \frac{1}{1 + t^6/5}\, dt = \frac{\sqrt{5}}{3 \cdot 5}\int \frac{du}{1 + u^2}\, dt = \frac{1}{3\sqrt{5}} \tan^{-1}(t^3/\sqrt{5}) + C.$

4. $\displaystyle\int \frac{\sin^{-1}(x)}{\sqrt{1 - x^2}}\, dx$

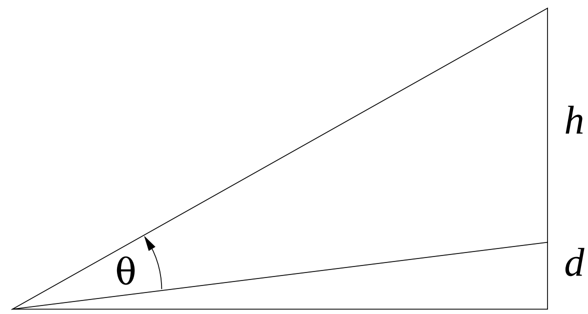
Putting $u = \sin^{-1}(x)$ and $du = \dfrac{1}{\sqrt{1 - x^2}}$, we have $\displaystyle\int \frac{\sin^{-1}(x)}{\sqrt{1 - x^2}}\, dx = \int u\, du = (\sin^{-1}(x))^2 + C.$

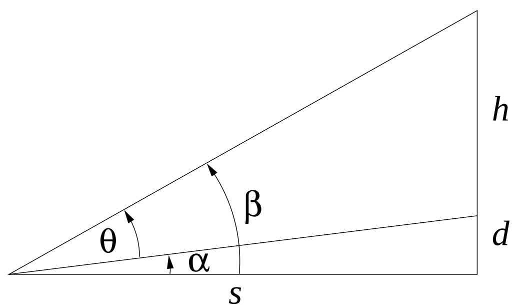
IX. (4) Solve for x in the equation $2^{ax} = \ln(c)\,3^{bx}$.

Applying the logarithm to both sides, we have:

$$\ln(2^{ax}) = \ln(\ln(c)\,3^{bx}) = \ln(\ln(c)) + \ln(3^{bx})$$
$$ax\ \ln(2) = \ln(\ln(c)) + bx\ \ln(3)$$
$$(a\ln(2) - b\ln(3))\,x = \ln(\ln(c))$$
$$x = \frac{\ln(\ln(c))}{(a\ln(2) - b\ln(3))}$$

X. (6) A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer. How far from the wall should the observer stand so as to get the best view (that is, so that the angle θ is largest)?





Letting s be the distance from the eye to the wall, we have $\theta = \beta - \alpha = \tan^{-1}\left(\frac{h+d}{s}\right) - \tan^{-1}\left(\frac{d}{s}\right)$. We calculate

$$\frac{d\theta}{ds} = \frac{1}{1+\left(\frac{h+d}{s}\right)^2}\left(-\frac{h+d}{s^2}\right) - \frac{1}{1+\left(\frac{d}{s}\right)^2}\left(-\frac{d}{s^2}\right) = -\frac{h+d}{s^2+(h+d)^2} + \frac{d}{s^2+d^2}$$

and find the critical point of θ:

$$\frac{h+d}{s^2+(h+d)^2} = \frac{d}{s^2+d^2}$$
$$(h+d)s^2 + (h+d)d^2 = ds^2 + d(h+d)^2$$
$$hs^2 = d(h+d)^2 - (h+d)d^2 = d(h+d)(h+d-d) = hd(h+d)$$
$$s^2 = d(h+d)$$
$$s = \sqrt{d(h+d)}$$

There is only one critical point, and θ is near 0 when s is close to 0 or when s is very large, so this must be the value of s that maximizes θ.