Mathematics 2423-001H Name (please print) ______________________

Examination I

February 13, 2007

Instructions: Give brief, clear answers. It is not expected that most people will be able to answer all the questions, just do what you can in 75 minutes. All integrals may be assumed to exist— there are no tricks involving functions that are not continuous on the domain of integration.

I. Determine the values of the following integrals:
(24)

1. $\displaystyle\int \sin(\pi t)\,dt$

   Putting $u = \pi t$ and $dt = (1/\pi)\,du$ gives $\displaystyle\int \sin(\pi t)\,dt = \int (1/\pi)\sin(u)\,du = -(1/\pi)\cos(u) + C = -(1/\pi)\cos(\pi t) + C$

2. $\displaystyle\int_{\pi/4}^{\pi/2} \csc^2(w)\,dw$

   $$\int_{\pi/4}^{\pi/2} \csc^2(w)\,dw = -\cot(w)\Big|_{\pi/4}^{\pi/2} = -\cot(\pi/2) - (-\cot(\pi/4)) = 0 - (-1) = 1$$

3. $\displaystyle\int \sec^5(\theta)\tan(\theta)\,d\theta$

   Putting $u = \sec(\theta)$ and $du = \sec(\theta)\tan(\theta)d\theta$ gives $\displaystyle\int \sec^5(\theta)\tan(\theta)\,d\theta = \int u^4\,du = u^5/5 + C = \sec^5(\theta)/5 + C$

4. $\displaystyle\int_0^3 t\sqrt{9-t^2}\,dt$

   Putting $u = 9 - t^2$ and $t\,dt = -(1/2)du$ gives $\displaystyle\int t\sqrt{9-t^2}\,dt = \int -(1/2)\sqrt{u}\,du = -(1/2)u^{3/2}/(3/2) + C = -(9-t^2)^{3/2}/3 + C$, consequently $\displaystyle\int_0^3 t\sqrt{9-t^2}\,dt = -(9-t^2)^{3/2}/3\Big|_0^3 = 0 - (-9^{3/2})/3 = 9$

5. $\displaystyle\int_0^3 \sqrt{9-t^2}\,dt$

   This is the area of the portion of the circle of radius 3 that lies in the first quadrant, so it equals $\pi \cdot 3^2/4 = 9\pi/4$.

6. $\displaystyle\int \frac{x}{\sqrt{x+2}}\,dx$

   Putting $u = x + 2$ and $dx = du$ gives $\displaystyle\int \frac{x}{\sqrt{x+2}}\,dx = \int \frac{u-2}{\sqrt{u}}\,du = \int \sqrt{u} - \frac{2}{\sqrt{u}}\,du = (2/3)u^{3/2} - 2u^{1/2}/(1/2) + C = (2/3)(x+2)^{3/2} - 4\sqrt{x+2} + C = (2(x+2) - 12)\sqrt{x+2}/3 = (2x-8)\sqrt{x+2}/3$

II. Calculate the following derivatives:
(6)

1. $\displaystyle\frac{d}{dx}\int_x^1 \sqrt{\tan(t)}\,dt$

   $$\frac{d}{dx}\int_x^1 \sqrt{\tan(t)}\,dt = \frac{d}{dx}\Big(-\int_1^x \sqrt{\tan(t)}\,dt\Big) = -\sqrt{\tan(x)}$$

2. $\dfrac{d}{dx}\displaystyle\int_{\tan(x)}^{1} \sqrt{\tan(t)}\,dt$

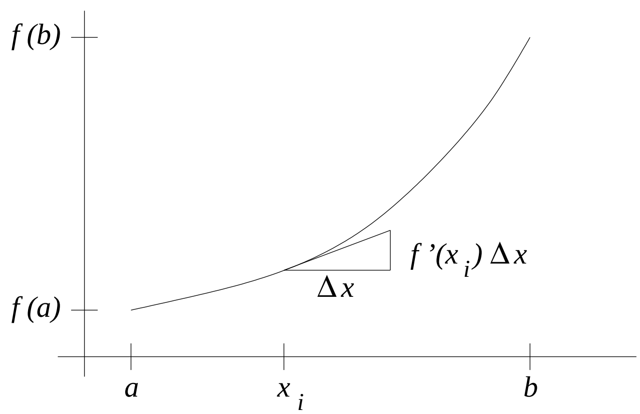
$$\frac{d}{dx}\int_{\tan(x)}^{1} \sqrt{\tan(t)}\,dt = \frac{d}{dx}\Big(-\int_{1}^{\tan(x)} \sqrt{\tan(t)}\,dt\Big) = \frac{d}{\tan(x)}\Big(-\int_{1}^{\tan(x)} \sqrt{\tan(t)}\,dt\Big)\frac{d(\tan(x))}{dx}$$
$$= -\sqrt{\tan(\tan(x))}\,\sec^2(x)$$

III. (5) Explain (briefly and concisely) the terms *partition* and *Riemann sum,* and how they are used to define the definite integral $\displaystyle\int_a^b f(x)\,dx$.

A partition is a way to break an interval into small intervals. In the context of integration, we typically break an interval $[a, b]$ into n equal subintervals, each of length $(b-a)/n$, which we call Δx. The i^{th} subinterval goes from $a + (i-1)\Delta x$ to $a + i\Delta x$. To form a Riemann sum for this partition, we choose any points, one from each subinterval. Writing x_i^* for the selected point in the i^{th} interval, the corresponding Riemann sum is $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$. The definite integral $\displaystyle\int_a^b f(x)\,dx$ is defined to be the limit of such sums as $n \to \infty$, if the limit exists.

IV. (3) Explain geometrically why the integral of the rate of change of a function should equal the total net change of a function, that is, why $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$. (This is, of course, the second assertion of the Fundamental Theorem of Calculus.)

Roughly speaking, a term $f'(x_i^*)\,\Delta x$ approximates the net change of f on the i^{th} interval, as suggested in this picture:



Thus a Riemann sum $\displaystyle\sum_{i=1}^{n} f'(x_i^*)\,\Delta x$ approximates the total net change $f(b) - f(a)$, so it is at least plausible that their limit, that is, $\displaystyle\int_a^b f'(x)\,dx$, equals $f(b) - f(a)$.

V. (6) As you know, the natural logarithm function is defined to be $\ln(x) = \int_1^x \frac{1}{t}\,dt$. In particular, if a and b are any numbers larger than 1, then $\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt$.

(a) Use the substitution $u = t/a$ to show that $\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{u}\,du = \ln(b)$.

Putting $u = t/a$ and $a\,du = dt$, we have $u = 1$ when $t = a$ and $u = b$ when $t = ab$, so $\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{au}\,a\,du = \int_1^b \frac{1}{u}\,du = \ln(b)$.

(b) Use part (a) and additivity of the integral on domains to verify that $\ln(ab) = \ln(a) + \ln(b)$.

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \ln(a) + \ln(b).$$

VI. (3) Find the value c that satisfies the Mean Value Theorem for Integrals for the function $f(x) = x^3$ on the interval $[0, 2]$.

We would need $f(c)\,(2-0) = \int_0^2 x^3\,dx = x^4/4\Big|_0^2 = 4$, so $c^3 = f(c) = 2$. Therefore $c = \sqrt[3]{2}$.

VII. (4) Let $g(x)$ be the function on the interval $[0, 10]$ defined by $g(x) = 0$ if x is rational, and $g(x) = 1$ if x is irrational. Partition $[0, 10]$ into two equal subintervals.

1. Calculate Δx.

   $\Delta x = (10-0)/2 = 5$.

2. Give three explicit choices of sample points x_1^* and x_2^* so that the Riemann sum $\sum_{k=1}^{2} g(x_i^*)\,\Delta x$ has different values for the three choices.

   Choosing both rational, say $x_1^* = 1$ and $x_2^* = 6$, makes the sum equal to $g(1)\cdot 5 + g(6)\cdot 5 = 0 + 0 = 0$. Choosing one irrational and the other rational, say $x_1^* = \pi$ and $x_2^* = 6$, makes the sum equal to $g(\pi)\cdot 5 + g(6)\cdot 5 = 5 + 0 = 5$. And choosing both irrational, say $x_1^* = \pi$ and $x_2^* = 2\pi$, makes the sum equal to $g(\pi)\cdot 5 + g(6)\cdot 5 = 5 + 5 = 10$.

VIII. (4) Use the telescoping sum $x^{n+1} - 1 = \sum_{k=1}^{n+1} x^k - x^{k-1}$ to verify that $(x^n + x^{n-1} + \cdots + x + 1)(x-1) = x^{n+1} - 1$.

$$x^{n+1} - 1 = \sum_{k=1}^{n+1} x^k - x^{k-1} = \sum_{k=1}^{n+1} (x-1)x^{k-1} = (x-1)\sum_{k=1}^{n+1} x^{k-1} = (x-1)(1 + x + x^2 + \cdots + x^n)$$