Mathematics 2423-001H | Name (please print) ____________________

Final Examination

May 8, 2007

I. (8) Consider the portion of the graph $y = e^{-x}$ between $x = 0$ and $x = 1$. For each of the following, write an integral whose value is the specified quantity for this portion of the graph, but *do not* attempt to evaluate the integrals.

1. The length of this portion of the graph.

$$ds = \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (-e^{-x})^2}\,dx = \sqrt{1 + e^{-2x}}\,dx, \text{ so the length is } \int_0^1 \sqrt{1 + e^{-2x}}\,dx$$

2. The surface area obtained when it is rotated about the x-axis.

The distance from (x, e^{-x}) to the x-axis is e^{-x}, so the surface area is $\displaystyle\int_0^1 2\pi e^{-x}\sqrt{1 + e^{-2x}}\,dx$.

3. The surface area obtained when it is rotated about the line $y = -1$.

The distance from (x, e^{-x}) to the line $y = -1$ is $1+e^{-x}$, so the surface area is $\displaystyle\int_0^1 2\pi(1+e^{-x})\sqrt{1 + e^{-2x}}\,dx$.

4. The surface area obtained when it is rotated about the y-axis.

The distance from (x, e^{-x}) to the y-axis is x, so the surface area is $\displaystyle\int_0^1 2\pi x\sqrt{1 + e^{-2x}}\,dx$.

II. (6) Simpson's Rule states that $\displaystyle\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{n-1} + y_n)$, with error of magnitude at most $\dfrac{K(b-a)}{180}h^4$, where $|f^{(4)}(x)| \le K$ for $a \le x \le b$. Use Simpson's rule with $n = 4$ to approximate $\displaystyle\int_{-1}^1 x^4\,dx$, and give a bound for the error. Leave both answers as fractions, not decimals.

We have $x_0 = -1$, $x_1 = -1/2$, $x_2 = 0$, $x_3 = 1/2$, and $x_4 = 1$, so $y_0 = 1$, $y_1 = 1/16$, $y_2 = 0$, $y_3 = 1/16$, and $y_4 = 1$. Also $h = (2-0)/4 = 1/2$. So the formula for Simpson's Rule gives the estimate

$$\frac{1/2}{3}\Big(1 + 4\cdot(1/16) + 2\cdot 0 + 4\cdot(1/16) + 1\Big) = \frac{5}{12}\ .$$

To calculate the error, we know that the fourth derivative of x^4 is the constant function $4! = 24$, whose maximum value on $[0, 2]$ is $K = 16$. Also, $h^4 = (1/2)^4 = 1/16$. So the error bound is

$$\frac{24\cdot(1-(-1))}{180\cdot 16} = \frac{1}{60}\ .$$

III. Let C be the portion of the unit circle that lies in the first quadrant.
(6)
(i) Write the standard equation for C of the form $y = f(x)$, $0 \le x \le 1$, and calculate that $ds = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

The standard equation is $y = \sqrt{1-x^2}$, and we calculate

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx = \sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}}\,dx = \sqrt{\frac{1}{1-x^2}}\,dx = \frac{1}{\sqrt{1-x^2}}\,dx$$

(ii) Integrate this to find the length of C. If the integral is improper, show the details of how you handle it.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1}\sin^{-1}(x)\Big|_0^b = \lim_{b\to 1}\sin^{-1}(b) - \sin^{-1}(0) = \frac{\pi}{2}$$

IV. Verify that $y = a\sinh(x) + b\cosh(x)$ is a solution to the differential equation $y'' = y$.
(3)

Calculating $y' = a\cosh(x) + b\sinh(x)$ and $y'' = a\sinh(x) + b\cosh(x)$, we have $y'' = y$.

V. State the Fundamental Theorem of Calculus (both parts, of course).
(6)

For a continuous function $f(x)$, $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$, and if $F'(x) = f(x)$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

VI. Calculate a Riemann sum for the function $f(x) = x^2$ on the interval $[0,6]$, using the partition with $x_1 = 1$,
(4) $x_2 = 2$, and $x_3 = 4$, and using midpoints as the sample points.

The endpoints are $x_0 = 0$ and $x_4 = 6$. We have $\Delta x_1 = x_1 - x_0 = 1$, $\Delta x_2 = x_2 - x_1 = 1$, $\Delta x_3 = x_3 - x_2 = 2$, and $\Delta x_4 = x_4 - x_3 = 2$. The midpoints are $x_1^* = 1/2$, $x_2^* = 3/2$, $x_3^* = 3$, and $x_4^* = 5$, so the Riemann sum is

$$f(x_1^*)\,\Delta x_1 + f(x_2^*)\,\Delta x_2 + f(x_3^*)\,\Delta x_3 + f(x_4^*)\,\Delta x_4 = \tfrac{1}{4}\cdot 1 + \tfrac{9}{4}\cdot 1 + 9\cdot 2 + 25\cdot 2 = 70\,\tfrac{1}{2}$$

VII. For each of the following rational functions, write out the precise *form* of the partial fraction decomposition.
(8) *Do not* solve for unknown values of the coefficients.

1. $\dfrac{x^5 - x^2}{(x^3+x)^3}$

The denominator factors into linear and irreducible quadratic factors as $x^3(x^2+1)^3$, so the decomposition is

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} + \frac{Hx+I}{(x^2+1)^3}$$

(Since $\dfrac{x^5-x^2}{(x^3+x)^3} = \dfrac{x^3-1}{x(x^2+1)^3}$, the terms $\dfrac{B}{x^2}$ and $\dfrac{C}{x^3}$ can be omitted.)

2. $\dfrac{1}{(x^2+x+1)(x^2+x-1)}$

x^2+x+1 is irreducible, but x^2+x-1 has real roots $\dfrac{-1\pm\sqrt{5}}{2}$, so it factors as $\left(x - \dfrac{-1-\sqrt{5}}{2}\right)\left(x - \dfrac{-1+\sqrt{5}}{2}\right)$. So the decomposition is

$$\frac{A}{x - \dfrac{-1-\sqrt{5}}{2}} + \frac{B}{x - \dfrac{-1+\sqrt{5}}{2}} + \frac{Cx+D}{x^2+x+1}$$

VIII. Use l'Hôpital's rule to evaluate the following limits.

(6)

1. $\lim_{x\to 0^+} \sin(x)\,\ln(x)$

$$\lim_{x\to 0^+} \sin(x)\,\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} = \lim_{x\to 0^+} \frac{1/x}{-\csc(x)\cot(x)}$$
$$= \lim_{x\to 0^+} \frac{\sin^2(x)}{-x\cos(x)} = \lim_{x\to 0^+} -\frac{\sin(x)}{\cos(x)}\,\frac{\sin(x)}{x} = (-0)\cdot 1 = 0$$

2. $\lim_{x\to 0} x^x$

$$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{\ln(x^x)} = \lim_{x\to 0+} e^{x\ln(x)} = \lim_{x\to 0+} e^{\ln(x)/(1/x)} = \lim_{x\to 0+} e^{(1/x)/(-1/x^2)} = \lim_{x\to 0+} e^{-x} = 1$$

IX. Evaluate the following integrals:

(20)

1. $\int \frac{\log_{10}(x)}{x}\,dx$

$\int \frac{\log_{10}(x)}{x}\,dx = \int \frac{1}{\ln(10)}\,\frac{\ln(x)}{x}\,dx$. Using $u = \ln(x),\ du = \frac{1}{x}\,dx$, the latter is $\frac{(\ln(x))^2}{2\,\ln(10)} + C$.

2. $\int \frac{\cosh(x)}{\cosh^2(x)-1}\,dx$

$\int \frac{\cosh(x)}{\cosh^2(x)-1}\,dx = \int \frac{\cosh(x)}{\sinh^2(x)}\,dx$. Using $u = \sinh(x)$, $du = \cosh(x)\,dx$, the latter is $\frac{(\sinh(x))^{-1}}{-1} + C = \frac{-1}{\sinh(x)} + C$.

3. $\int_1^{\ln(5)} x^2\,e^x\,dx$

Use parts twice. The first time, $u = x^2$, $du = 2x\,dx$, $dv = e^x\,dx$, and $v = e^x$, so $\int_1^{\ln(5)} x^2\,e^x\,dx = 2x\,e^x\Big|_1^{\ln(5)} - \int_1^{\ln(5)} x^2\,e^x\,dx = (\ln(5))^2\cdot 5 - 2\cdot 1 - \int_1^{\ln(5)} 2x\,e^x\,dx = 5(\ln(5))^2 - 2 - \int_1^{\ln(5)} 2x\,e^x\,dx$. Now using parts with $u = 2x$, $du = 2\,dx$, $dv = e^x\,dx$, and $v = e^x$, the latter expression is $5(\ln(5))^2 - 2 - \int_1^{\ln(5)} 2x\,e^x\,dx = 5(\ln(5))^2 - 2 - \left(2x\,e^x\Big|_1^{\ln(5)} - \int_1^{\ln(5)} e^x\,dx\right) = 5(\ln(5))^2 - 2 - \Big(2\ln(5)\cdot 5 - 2\cdot 1 - (5-e)\Big) = 5(\ln(5))^2 - 10\ln(5) + 5 - e$.

4. $\int \frac{1}{x^2+x+1}\,dx$

Completing the square and substituting $u = \frac{2x+1}{\sqrt{3}}$, we have

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+x+1/4+3/4}\,dx = \int \frac{1}{(x+1/2)^2+3/4}\,dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}\,dx$$
$$= \frac{\sqrt{3}}{2}\,\frac{4}{3}\int \frac{1}{1+u^2}\,du = \frac{2}{\sqrt{3}}\,\tan^{-1}(u) + C = \frac{2}{\sqrt{3}}\,\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

5. $\displaystyle\int x\,\cos^2(x)\,dx$

$$\int x\,\cos^2(x)\,dx = \int x\left(\tfrac{1}{2}+\tfrac{1}{2}\cos(2x)\right)dx = \int \tfrac{x}{2}+\tfrac{x}{2}\cos(2x)\,dx = \tfrac{x^2}{4}+\int \tfrac{x}{2}\cos(2x)\,dx.$$

Using parts with $u=\frac{x}{2}$ and $dv=\cos(2x)\,dx$, the latter expression becomes

$$\tfrac{x^2}{4}+\tfrac{x}{4}\sin(2x)-\int \tfrac{1}{4}\sin(2x)\,dx = \tfrac{x^2}{4}+\tfrac{x}{4}\sin(2x)+\tfrac{\cos(2x)}{8}+C.$$

X. Find the domain and range of the function $f(x)=\ln(\tan^{-1}(x))$.

(4)

In order that the logarithm be defined, we must have $\tan^{-1}(x)>0$. This is true exactly when $x>0$, so the domain is $(0,\infty)$. The values of $\tan^{-1}(x)$ when x is in $(0,\infty)$ are $(0,\frac{\pi}{2})$, so the values of $\ln(\tan^{-1}(x))$ lie between $-\infty$ and $\ln(\frac{\pi}{2})$. That is, the range is $(-\infty,\ln(\frac{\pi}{2}))$.

XI. Consider the function $y=e^{-x}$.

(8)

1. Calculate ds.

   We first calculate that $ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+(-e^{-x})^2}\,dx=\sqrt{1+e^{-2x}}\,dx$.

2. Write an improper integral whose value is the surface area produced when the graph of the function $y=e^{-x}$, $0\le x<\infty$, is rotated about the x-axis.

   The distance to the axis of rotation is e^{-x}, so the integral is $\displaystyle\int_0^\infty 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx$.

3. Evaluate the integral, using the substitution $u=e^{-x}$ and the integration formula $\displaystyle\int\sqrt{a^2+u^2}\,du=\frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln(u+\sqrt{a^2+u^2})+C$.

   Using the substitution $u=e^{-x}$, $du=-e^{-x}\,dx$, and noting that $u\to 0$ as $x\to\infty$, we calculate

$$\begin{aligned}\int_0^\infty 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx &= \lim_{b\to\infty}\int_0^b 2\pi e^{-x}\sqrt{1+e^{-2x}}\,dx = \lim_{b\to 0}\int_1^b -2\pi\sqrt{1+u^2}\,du\\ &= \lim_{b\to 0} -2\pi\frac{u}{2}\sqrt{1+u^2}-2\pi\,\frac{1}{2}\,\ln(u+\sqrt{1+u^2})\Big|_1^b\\ &= \lim_{b\to 0} -\pi b\sqrt{1+b^2}-\pi\,\ln(b+\sqrt{1+b^2})+\pi\sqrt{1+1}+\pi\,\ln(1+\sqrt{1+1})\\ &= 0-\pi\,\ln(1)+\sqrt{2}\,\pi+\ln(1+\sqrt{2})\,\pi=\sqrt{2}\,\pi+\ln(1+\sqrt{2})\,\pi\end{aligned}$$