Instructions: Give brief, clear answers. “Prove” means “give an argument”.

I. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$. Prove that if $f$ and $g$ are injective, then the composition $g \circ f$ is injective. (4)

II. State the Fundamental Theorem of Arithmetic. (4)

III. Prove that if $a \equiv b \mod m$ and $b \equiv c \mod m$, then $a \equiv c \mod m$. (4)

IV. Give Euclid’s proof that there are infinitely many primes. (4)

V. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever $n$ is a positive integer. (5)

VI. (a) Show that $ac \equiv bc \mod m$ and $c \not\equiv 0 \mod m$ does not always imply that $a \equiv b \mod m$. (4)

(b) Tell without proof a condition (which always holds when $m$ is prime and $c \not\equiv 0 \mod m$) that ensures that $ac \equiv bc \mod m$ does imply that $a \equiv b \mod m$. (4)

VII. Let $X$ be the set of all infinite sequences in which each term is one of the letters $x$, $y$, or $z$. Some elements of $X$ are $yyyyyyyyyyyy\cdots$, $xxyzzxxxxyyzzz\cdots$, and $xxyyzzzxyzzzyzzzyxzxyyxxzyxzyxzyyzzzzzzz\cdots$. Using Cantor’s idea, prove that there does not exist any surjective function from $\mathbb{N}$ to $X$. (5)

VIII. Let $a$, $b$, and $c$ be integers. Using the definition of “divides”, prove that if $a|b$ and $b|c$, then $a|c$. (4)

IX. Let $Y$ be the set of all positive fractions (not rational numbers, so $\frac{1}{2}$ and $\frac{2}{4}$ are different fractions). Using Cantor’s idea, prove that $Y$ is countable. (4)

X. Let $A$ be a nonempty set, so that we can choose an element $a_0$ of $A$. Prove that there exists a surjective function from $\mathcal{P}(A)$ to $A$. (4)

XI. Let $m$ and $n$ be two positive integers. Show that if $mn = 360$ and the least common multiple of $m$ and $n$ is 10 times their greatest common divisor, then both $m$ and $n$ are divisible by 6. (4)

XII. Let $Z$ be an infinite set.

(a) Informally, saying that $Z$ is countable means that it is possible to list the elements of $Z$. This is not a real definition, since the word “list” is not precise. Give the formal definition of “$Z$ is countable.”

(b) Now suppose that $Z$ is set of all infinite sequences in which each term is one of the letters a, or b, and exactly one of the terms is a. Some elements of $Y$ are abbbbbbb\cdots, bbbbbbbbbbbbbbb\cdots, and bbbbbbb\cdotsbbbbbb\cdots, where in the last sequence the a appears after exactly 35,014,227 b’s have appeared. Prove that $Z$ is countable. (5)