Instructions: Give brief, clear answers. “Prove” means “give an argument”. In giving definitions, give the precise definition, using logical notation and/or set notation as appropriate.

I. Write the following as an implication: “$b^2 \geq 2$ for at most one $b$”.

$$(b^2 \geq 2 \land c^2 \geq 2) \Rightarrow b = c$$

II. Let $T(p, c)$ be “Person $p$ has traveled to the city $c$.” Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets $\mathcal{P}$ of all people and $\mathcal{C}$ of all destination cities. If your answer involves a negation, simplify as much as possible.

(a) Joan has been to Paris or London.

$$T(\text{Joan}, \text{Paris}) \lor T(\text{Joan}, \text{London})$$

(b) Everyone has traveled to at least one city.

$$\forall p \in \mathcal{P}, \exists c \in \mathcal{C}, T(p, c),$$

(c) No one has traveled to every city.

$$\neg \exists p \in \mathcal{P}, \forall c \in \mathcal{C}, T(p, c),$$

which simplifies to

$$\forall p \in \mathcal{P}, \exists c \in \mathcal{C}, \neg T(p, c),$$

(d) Any two people have traveled to at least one city in common.

$$\forall p \in \mathcal{P}, \forall q \in \mathcal{P}, \exists c \in \mathcal{C}, (T(p, c) \land T(q, c))$$

III. For the function $H: s \to t$ (where $s$ and $t$ are sets), give definitions of the following. As requested in the instructions, give the precise definitions, using logical notation and/or set notation as appropriate.

(a) the range of $H$

$$\{H(x) \mid x \in s\}$$

(b) the preimage of an element $T$ of $t$

$$\{x \in s \mid H(x) = T\}$$

(c) the inverse function $H^{-1}$, assuming that $N$ is bijective (part of giving the definition of a function is telling its domain and codomain).

$$H^{-1}: t \to s \text{ is defined by } H^{-1}(y) = x \iff H(x) = y$$

(d) the composition $G \circ H$, assuming that $G: t \to u$ (part of giving the definition of a function is telling its domain and codomain).

$$G \circ H: s \to u \text{ is defined by } G \circ H(x) = G(H(x))$$

(e) $H = K$, where $K: u \to v$

$$H = K \text{ when } s = u, t = v, \text{ and } \forall x \in s, H(x) = K(x)$$

(f) the graph of $H$

$$\{(x, H(x)) \mid x \in s\}, \text{ a subset of } s \times t.$$
IV. Let $X$ be an infinite set.
(a) Define what it means to say that $X$ is countable.

$X$ is countable where there exists a bijective function from $\mathbb{N}$ to $X$.

(b) Show a function that verifies that $\mathbb{Z}$ is countable.

A bijective function from $\mathbb{N}$ to $\mathbb{Z}$ is given by using the pattern

$1 \mapsto 0,$
$2 \mapsto 1, 3 \mapsto -1,$
$4 \mapsto 2, 5 \mapsto -2,$
$6 \mapsto 3, 7 \mapsto -3,$
$8 \mapsto 4, 9 \mapsto -4,$

and so on.

V. Write out all elements of $\mathcal{P}(\{1, 2\})$ and all elements of $\mathcal{P}(\{1, 2\}) \times \{1, 2, 3\}$.

$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

$\mathcal{P}(\{1, 2\}) \times \{1, 2, 3\} = \{(\emptyset, 1), (\emptyset, 2), (\emptyset, 3), (\{1\}, 1), (\{1\}, 2), (\{1\}, 3), (\{2\}, 1), (\{2\}, 2), (\{2\}, 3), (\{1, 2\}, 1), (\{1, 2\}, 2), (\{1, 2\}, 3)\}$

VI. Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(m, n) = m^2 - n$. Prove that $f$ is surjective.

Let $k \in \mathbb{Z}$. Then, $f(0, -k) = 0^2 - (-k) = k$.

VII. Let $f: X \to Y$ and $g: Y \to Z$. Prove that if $f$ and $g$ are injective, then the composition $g \circ f$ is injective.

Assume that $f$ and $g$ are injective. Let $x_1, x_2 \in X$ and assume that $g \circ f(x_1) = g \circ f(x_2)$. This says that $g(f(x_1)) = g(f(x_2))$. Since $g$ is injective, this implies that $f(x_1) = f(x_2)$. Since $f$ is injective, this implies that $x_1 = x_2$.

VIII. Disprove the following assertion: for all sets $A$, $B$, and $C$, if $A \cap C = B \cap C$, then $A = B$.

Put $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{1\}$. Then $A \cap C = \{1\}$ and $B \cap C = \{1\}$, but $A \neq B$.

IX. Let $f: (0, \pi) \to (0, \infty)$ be the function defined by $f(x) = \csc(x)$, where the cosecant function is as usual given by $\csc(x) = \frac{1}{\sin(x)}$.

(a) Prove that $f$ is not injective.

$f(\pi/4) = \frac{1}{\sqrt{2}} = \sqrt{2}$ and $f(3\pi/4) = \frac{1}{\sqrt{2}} = \sqrt{2}$, but $\pi/4 \neq 3\pi/4$.

(b) Prove that $f$ is not surjective.

Consider $1/2$, an element of the codomain $(0, \infty)$. For all $x \in (0, \pi)$, $0 < \sin(x) \leq 1$, so $1 \leq \csc(x)$. Therefore $\csc(x) \neq 1/2$.

(Alternatively, we can use proof by contradiction: Suppose for contradiction that there exists $x \in (0, \pi)$ such that $\csc(x) = 1/2$. Then $1/\sin(x) = 1/2$ so $\sin(x) = 2$, contradicting the fact that $-1 \leq \sin(x) \leq 1$ for all $x$.)

X. Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is injective but not surjective.

The exponential function $e^x$ and the inverse tangent function $\tan^{-1}(x)$ are perhaps the most familiar of many possible examples.
XI. Let $A = \mathbb{R}$ and $B = \mathbb{Z}$. Give examples of each of the following.

(a) An element of $A \times B$ that is not in $B \times B$.
   
   $(1/2, 1)$

(b) An element of $B \times A$ that is not in $B \times B$.
   
   $(1, 1/2)$

(c) An element of $A \times A$ that is neither in $A \times B$ nor in $B \times A$.
   
   $(1/2, 1/2)$