Mathematics 2513-001 Name (please print) ____________________

Examination II Form A

March 23, 2006

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Instructions: Give brief, clear answers. "Prove" means "give an argument". In giving definitions, give the *precise* definition, using logical notation and/or set notation as appropriate.

**I.** (2) Write the following as an implication: "$a^2 \geq 2$ for at most one $a$".

$$(a^2 \geq 2 \wedge b^2 \geq 2) \Rightarrow a = b$$

**II.** (4) Let $T(p, c)$ be "Person $p$ has traveled to the city $c$." Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets $\mathcal{P}$ of all people and $\mathcal{C}$ of all destination cities. If your answer involves a negation, simplify as much as possible.

(a) Jeff has been to Madrid or Paris.

$T(\text{Jeff, Madrid}) \vee T(\text{Jeff, Paris})$

(b) No one has traveled to every city.

$\neg\exists p \in \mathcal{P}, \forall c \in \mathcal{C}, T(p, c)$, which simplifies to $\forall p \in \mathcal{P}, \exists c \in \mathcal{C}, \neg T(p, c)$,

(c) Everyone has traveled to at least one city.

$\forall p \in \mathcal{P}, \exists c \in \mathcal{C}, T(p, c)$,

(d) Any two people have traveled to at least one city in common.

$\forall p \in \mathcal{P}, \forall q \in \mathcal{P}, \exists c \in \mathcal{C}, (T(p, c) \wedge T(q, c))$

**III.** (4) Write out all elements of $\mathcal{P}(\{a, b\})$ and all elements of $\mathcal{P}(\{a, b\}) \times \{a, b, c\}$.

$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$\mathcal{P}(\{a, b\}) \times \{a, b, c\} = \{(\emptyset, a), (\emptyset, b), (\emptyset, c), (\{a\}, a), (\{a\}, b), (\{a\}, c), (\{b\}, a), (\{b\}, b), (\{b\}, c), (\{a, b\}, a), (\{a, b\}, b), (\{a, b\}, c)\}$

**IV.** (10) For the function $G\colon s \to t$ (where $s$ and $t$ are sets), give definitions of the following. As requested in the instructions, give the *precise* definitions, using logical notation and/or set notation as appropriate.

(a) the range of $G$

$\{G(x) \mid x \in s\}$

(b) the preimage of an element $T$ of $t$

$\{x \in s \mid G(x) = T\}$

(c) the inverse function $G^{-1}$, assuming that $G$ is bijective (part of giving the definition of a function is telling its domain and codomain).

$G^{-1}\colon t \to s$ is defined by $G^{-1}(y) = x \Leftrightarrow G(x) = y$

(d) the composition $H \circ G$, assuming that $H\colon t \to u$ (part of giving the definition of a function is telling its domain and codomain).

$H \circ G\colon s \to u$ is defined by $H \circ G(x) = H(G(x))$

(e) $G = K$, where $K \colon u \to v$

$G = K$ when $s = u$, $t = v$, and $\forall x \in s, G(x) = K(x)$

(f) the graph of $G$

$\{(x, G(x)) \mid x \in s\}$, a subset of $s \times t$.

**V.** Let $X$ be an infinite set.
(4) (a) Define what it means to say that $X$ is *countable.*

$X$ is *countable* where there exists a bijective function from $\mathbb{N}$ to $X$.

(b) Show a function that verifies that $\mathbb{Z}$ is countable.

A bijective function from $\mathbb{N}$ to $\mathbb{Z}$ is given by using the pattern
$1 \mapsto 0$,
$2 \mapsto 1$, $3 \mapsto -1$,
$4 \mapsto 2$, $5 \mapsto -2$,
$6 \mapsto 3$, $7 \mapsto -3$,
$8 \mapsto 4$, $9 \mapsto -4$,
and so on.

**VI.** Let $f \colon X \to Y$ and $g \colon Y \to Z$. Prove that if $f$ and $g$ are injective, then the composition $g \circ f$ is injective.
(4)

Assume that $f$ and $g$ are injective. Let $x_1, x_2 \in X$ and assume that $g \circ f(x_1) = g \circ f(x_2)$. This says that $g(f(x_1)) = g(f(x_2))$. Since $g$ is injective, this implies that $f(x_1) = f(x_2)$. Since $f$ is injective, this implies that $x_1 = x_2$.

**VII.** Let $f \colon (0, \pi) \to (0, \infty)$ be the function defined by $f(x) = \csc(x)$, where the cosecant function is as usual
(8) given by $\csc(x) = \dfrac{1}{\sin(x)}$.

(a) Prove that $f$ is not injective.

$f(\pi/4) = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$ and $f(3\pi/4) = \dfrac{1}{1/\sqrt{2}} = \sqrt{2}$, but $\pi/4 \neq 3\pi/4$.

(b) Prove that $f$ is not surjective.

Consider $1/2$, an element of the codomain $(0, \infty)$. For all $x \in (0, \pi)$, $0 < \sin(x) \leq 1$, so $1 \leq \csc(x)$. Therefore $\csc(x) \neq 1/2$.

(Alternatively, we can use proof by contradiction: Suppose for contradiction that there exists $x \in (0, \pi)$ such that $\csc(x) = 1/2$. Then $1/\sin(x) = 1/2$ so $\sin(x) = 2$, contradicting the fact that $-1 \leq \sin(x) \leq 1$ for all $x$.)

**VIII.** Let $f \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(m, n) = m^2 - n$. Prove that $f$ is surjective.
(4)

Let $k \in \mathbb{Z}$. Then, $f(0, -k) = 0^2 - (-k) = k$.

**IX.** Disprove the following assertion: for all sets $A$, $B$, and $C$, if $A \cap B = A \cap C$, then $B = C$.
(3)

Put $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$. Then $A \cap B = \{1\}$ and $A \cap C = \{1\}$, but $B \neq C$.

**X.** Give an example of a function from $\mathbb{R}$ to $\mathbb{R}$ that is injective but not surjective.
(4)

The exponential function $e^x$ and the inverse tangent function $\tan^{-1}(x)$ are perhaps the most familiar of many possible examples.

**XI**. Let $A = \mathbb{R}$ and $B = \mathbb{Z}$. Give examples of each of the following.

(3) (a) An element of $A \times B$ that is not in $B \times B$.

$(1/2, 1)$

(b) An element of $B \times A$ that is not in $B \times B$.

$(1, 1/2)$

(c) An element of $A \times A$ that is neither in $A \times B$ nor in $B \times A$.

$(1/2, 1/2)$