Mathematics 2513-001 Name (please print) ______________________

Examination I Form A

February 14, 2006

Instructions: Give brief, clear answers.

**I.** (2) Write the following as an implication: "$a^2 \geq 2$ for at most one $a$".

$$(a^2 \geq 2 \wedge b^2 \geq 2) \Rightarrow a = b$$

**II.** (3) Using step-by-step logic, simplify the following expression: $\neg(\neg R \wedge \exists z, (\ Q(z) \Rightarrow P(z)\ )$

$$\neg(\neg R \wedge \exists z, (\ Q(z) \Rightarrow P(z)\ ) \equiv R \vee \neg\exists z, (\ Q(z) \Rightarrow P(z)\ )$$
$$\equiv R \vee \forall z, \neg(\ Q(z) \Rightarrow P(z)\ ) \equiv R \vee \forall z, \neg(\ \neg Q(z) \vee P(z)\ )$$
$$\equiv R \vee \forall z, (\ Q(z) \wedge \neg P(z)\ )$$

**III.** (4) Give the general form of a proof by contradiction. That is, if the statement to be proven is $P$, give the main steps in the logical structure of the proof. Briefly explain why the argument proves the original assertion.

The general form is:

*Statement:* $P$.
*proof:* Assume $\neg P$.
...
Therefore $Q$.
But $Q$ is false.
Therefore $P$. □

The first section of the argument proves the implication $\neg P \Rightarrow Q$, by a direct argument. This is equivalent to its contrapositive, $\neg Q \Rightarrow P$. Then, one observes that $\neg Q$ is true, so the implication $\neg Q \Rightarrow P$ guarantees that $P$ is true.

**IV.** (9) In this problem, you may take as known the fact that $\sqrt{2}$ is irrational.

(a) Prove that the difference of two rational numbers must be rational (that is, that if $x$ and $y$ are rational, then $x - y$ is rational).

Let $x$ and $y$ be rational numbers. Then, we can write $x = \dfrac{p}{q}$ and $y = \dfrac{r}{s}$ for some integers $p$, $q$, $r$, and $s$. We calculate that $x - y = \dfrac{ps - rq}{qs}$, so $x - y$ is also rational.

(b) Prove that the sum of a rational number and an irrational number must be irrational.

Suppose for contradiction that there exist a rational number $x$ and an irrational number $y$ for which $x + y$ is rational. Using part (a), $y = (x + y) - x$ is rational, which is a contradiction.

(c) Give a counterexample to: The difference of two irrational numbers must be irrational.

Put $x = \sqrt{2}$ and $y = \sqrt{2}$. Then $x - y = 0$, which is rational.

**V.** (3) Write the following statement in logical notation (and simplified so that it does not involve the negation symbol $\neg$) using the universal set $\mathcal{U} = \mathbb{Z}$: There is a positive integer that is not the sum of the squares of three integers.

$$\begin{aligned} &\exists n, (n > 0 \land \neg(\exists a, \exists b, \exists c, a^2 + b^2 + c^2 = n)) \\ \equiv\ &\exists n, (n > 0 \land \forall a, \forall b, \forall c, \neg(a^2 + b^2 + c^2 = n)) \\ \equiv\ &\exists n, (n > 0 \land \forall a, \forall b, \forall c, a^2 + b^2 + c^2 \neq n) \end{aligned}$$

**VI.** (3) Write each of the following as either $A \Rightarrow B$ or $B \Rightarrow A$:

(i) $A$ is necessary for $B$

$B \Rightarrow A$

(ii) $A$, when $B$

$B \Rightarrow A$

(iii) whenever $A$, $B$

$A \Rightarrow B$

**VII.** (5) Let $M(p, m)$ be "Person $p$ has seen the movie $m$." Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets $\mathcal{P}$ of all people and $\mathcal{M}$ of all movies. If your answer involves a negation, simplify as much as possible.

(a) Jeff has seen every movie.

$\forall m \in \mathcal{M}, M(\text{Jeff}, m)$

(b) Jack has never seen a movie.

$\neg\exists m \in \mathcal{M}, M(\text{Jack}, m)$, which simplifies slightly to $\forall m \in \mathcal{M}, \neg M(\text{Jack}, m)$.

(c) Mary has seen every movie that Fred has seen.

$\forall m \in \mathcal{M}, (\ M(\text{Fred}, m) \Rightarrow M(\text{Mary}, m)\ )$

(d) Everyone has seen at least one movie.

$\forall p \in \mathcal{P}, \exists m \in \mathcal{M}, M(p, m)$

(e) Between the two of them, Ellen and Max have seen every movie.

$\forall m \in \mathcal{M}, (\ M(\text{Ellen}, m) \lor M(\text{Max}, m)\ )$

**VIII.** (4) Use a truth table to verify the tautology $\big(\neg Z \Rightarrow (X \land \neg X)\big) \Rightarrow Z$.

| $Z$ | $X$ | $\neg X$ | $X \land \neg X$ | $\neg Z$ | $\neg Z \Rightarrow (X \land \neg X)$ | $\big(\neg Z \Rightarrow (X \land \neg X)\big) \Rightarrow Z$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | T | T |
| T | F | T | F | F | T | T |
| F | T | F | F | T | F | T |
| F | F | T | F | T | F | T |

**IX.** (8) Assuming that the universal set is $\mathcal{U} = \mathbb{R}$, prove (if the statement is true) or disprove (if the statement is false) each of the following statements.

1. $\forall x, (x > 0 \Rightarrow x > 1)$

   Counterexample: Putting $x = \frac{1}{2}$, we have $\frac{1}{2} > 0$ but $\frac{1}{2} \leq 1$.

2. $\exists x, (x > 0 \Rightarrow x > 1)$

   Proof: Put $x = 2$. Then $2 > 0$ and $2 > 1$, so $2 > 0 \Rightarrow 2 > 1$.

   A more amusing Proof: Put $x = -1$. Then $-1 > 0$ and $-1 > 1$ are false, so $-1 > 0 \Rightarrow -1 > 1$ is true.

3. $\forall x, (x > 1 \Rightarrow x > 0)$

   Proof: Let $x$ be a real number. Assume that $x > 1$. Since $1 > 0$, it follows that $x > 0$.

4. $\exists x, (x > 1 \Rightarrow x > 0)$

   Proof: Put $x = 2$. Then $2 > 1$ and $2 > 0$, so $2 > 1 \Rightarrow 2 > 0$.

**X.** (3) Assuming that the universal set is $\mathcal{U} = \mathbb{R}$, prove the statement $\forall x, \exists y, x > y$.

Let $x$ be a real number. Putting $y = x - 1$, we have $x > x - 1 = y$.

**XI.** (6) This problem concerns the following statement about integers: "If $5n + 4$ is even, then $n$ is even."

(a) Prove the statement by arguing the contrapositive.

We will argue the contrapositive. Assume that $n$ is odd. Then we can write $n = 2k + 1$ for some integer $k$. We calculate that $5n + 4 = 10k + 9 = 2(5k + 4) + 1$, so $5n + 4$ is odd.

(b) Prove the statement using proof by contradiction.

Suppose for contradiction that there is an integer $n$ for which $5n + 4$ is even, but $n$ is odd. We can write $n = 2k + 1$ for some integer $k$, and calculate that $5n + 4 = 10k + 9 = 2(5k + 4) + 1$. Therefore, $5n + 4$ must be odd, contradicting the assumption that it is even.