Mathematics 2513-001 Name (please print) ______________________

Examination I Form A

February 14, 2006

Instructions: Give brief, clear answers.

I. (2) Write the following as an implication: "$a^2 \geq 2$ for at most one a".

II. (3) Using step-by-step logic, simplify the following expression: $\neg(\neg R \wedge \exists z, (\, Q(z) \Rightarrow P(z)\,)$

III. (4) Give the general form of a proof by contradiction. That is, if the statement to be proven is P, give the main steps in the logical structure of the proof. Briefly explain why the argument proves the original assertion.

IV. (9) In this problem, you may take as known the fact that $\sqrt{2}$ is irrational.
(a) Prove that the difference of two rational numbers must be rational (that is, that if x and y are rational, then $x - y$ is rational).
(b) Prove that the sum of a rational number and an irrational number must be irrational.
(c) Give a counterexample to: The difference of two irrational numbers must be irrational.

V. (3) Write the following statement in logical notation (and simplified so that it does not involve the negation symbol $\neg$) using the universal set $\mathcal{U} = \mathbb{Z}$: There is a positive integer that is not the sum of the squares of three integers.

VI. (3) Write each of the following as either $A \Rightarrow B$ or $B \Rightarrow A$:
(i) A is necessary for B
(ii) A, when B
(iii) whenever A, B

VII. (5) Let $M(p, m)$ be "Person p has seen the movie m." Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets $\mathcal{P}$ of all people and $\mathcal{M}$ of all movies. If your answer involves a negation, simplify as much as possible.
(a) Jeff has seen every movie.
(b) Jack has never seen a movie.
(c) Mary has seen every movie that Fred has seen.
(d) Everyone has seen at least one movie.
(e) Between the two of them, Ellen and Max have seen every movie.

VIII. (4) Use a truth table to verify the tautology $\big(\neg Z \Rightarrow (X \wedge \neg X)\big) \Rightarrow Z$.

IX. (8) Assuming that the universal set is $\mathcal{U} = \mathbb{R}$, prove (if the statement is true) or disprove (if the statement is false) each of the following statements.

1. $\forall x, (x > 0 \Rightarrow x > 1)$
2. $\exists x, (x > 0 \Rightarrow x > 1)$
3. $\forall x, (x > 1 \Rightarrow x > 0)$
4. $\exists x, (x > 1 \Rightarrow x > 0)$

X. (3) Assuming that the universal set is $\mathcal{U} = \mathbb{R}$, prove the statement $\forall x, \exists y, x > y$.

XI. (6) This problem concerns the following statement about integers: "If $5n + 4$ is even, then n is even."
(a) Prove the statement by arguing the contrapositive.
(b) Prove the statement using proof by contradiction.