

Instructions: Give brief, clear answers. “Prove” means “give an argument”.

- I.** Let  $T(s, c)$  be “Student  $s$  has taken course  $c$ .” Write each of the following statements in logical notation, putting in all necessary quantifiers using the sets  $\mathcal{S}$  of all students and  $\mathcal{C}$  of all courses. If your answer involves a negation, simplify as much as possible.
- (5)
- (a) Ann has taken both Diffy Q and Linear.
  - (b) Everyone has taken English Comp.
  - (c) No student has taken every course.
  - (d) Someone (some one person) has taken all the courses that I have taken.
  - (e) Phillip has not taken any of the courses that Ann has.
- II.** Assuming that  $y: A \rightarrow B$ , give precise definitions of the following, using logical notation and/or set notation as appropriate. Remember that the domain and codomain are part of the definition of a function, so must be specified when you are defining a function.
- (8)
- (a) the range of  $y$
  - (b) the preimage of an element  $b$  of  $B$
  - (c) the inverse function  $y^{-1}$ , assuming that  $y$  is bijective
  - (d)  $y = x$
  - (e) the graph of  $y$
- III.** Define  $\gcd(a, b)$ . Define *relatively prime*. Explain why the integer 1 is relatively prime to every other integer.
- (4)
- IV.** Use proof by contradiction to prove that the sum of a rational number and an irrational number must be irrational.
- (4)
- V.** Tell how many strings of six letters satisfy each of the following conditions. Make reasonable simplifications, but leave products in factored form (that is, do not multiply them out).
- (10)
- (a) contain no repeated letter.
  - (b) start with  $z$  or end with  $z$ , but do not both start and end with  $z$ .
  - (c) contain exactly one vowel
  - (d) contain no immediate repeat (for example, no “bb”), although a letter can repeat later (“bab” can appear).
  - (e) have consonants in the first, third, and fifth position and vowels in the second, fourth, and sixth positions, and have no repeated consonant, although they may contain repeated vowels.
- VI.** State the (basic, not generalized) Pigeonhole Principle.
- (3)
- VII.** Prove that the function  $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  defined by  $f(x, y) = (2x + y, x + y)$  is injective.
- (4)
- VIII.** Determine the number of elements in each of the following sets. If binomial coefficients appear, write them as quotients of factorials, but do not multiply out the factorials.
- (8)
- (a)  $\mathcal{P}(\{1, 2\} \times \{1, 3\})$
  - (b)  $\mathcal{P}(\{\{2\} \cup \mathcal{P}(\{1, 2\} \times \{1, 3\})\})$
  - (c)  $\{S \subseteq \{1, 2, 3, \dots, 50\} \mid S \text{ has cardinality } 2\}$
  - (d)  $\{s \mid s \text{ is a bit string of length } 20 \text{ containing either exactly five } 0\text{'s or exactly fifteen } 0\text{'s}\}$  (recall that a bit string is a finite sequence of 0's and 1's).

- IX.** Write the following as an implication: “ $x^2 \geq 2$  for at most one  $x$ ”.
- (2)
- X.** Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ . Prove that if  $f$  and  $g$  are surjective, then the composition  $g \circ f$  is surjective.
- (4)
- XI.** Prove that  $2^n < n!$  whenever  $n \geq 4$ .
- (4)
- XII.** Let  $X$  be the set of all infinite sequences in which each term is an integer, that is, all sequence  $z_1 z_2 z_3 \cdots$ , where each  $z_i \in \mathbb{Z}$ . Using Cantor’s idea, prove that  $X$  is not countable.
- (4)
- XIII.** Let  $a, b, c, m$ , and  $n$  be integers.
- (6)
- (a) Using the definition of “divides”, prove that if  $a|b$  and  $a|c$ , then  $a|b + c$ .
- (b) Using the definition of “divides”, prove that if  $a|b$ , then  $a|mb$ .
- (c) Give a step-by-step argument using (a) and (b) to deduce: If  $a|b$  and  $a|c$ , then  $a|mb + nc$ .
- XIV.** Let  $A$  be a countable set, so that  $A$  can be written as  $\{a_1, a_2, a_3, \dots\}$ , and let  $B = \{b_1, b_2, b_3, \dots\}$  be another countable set.
- (4)
- (a) Write down a list (at least, the first few elements of such a list) of all elements of  $A \times B$  whose first coordinate is  $a_1$ .
- (b) Prove that  $A \times B$  is countable.
- XV.** State the Fundamental Theorem of Arithmetic.
- (2)
- XVI.** How many subsets of a set with 300 elements contain more than one element?
- (3)