Mathematics 2443-006H Name (please print) ____________________

Examination II

March 28, 2006

Instructions: Give brief, clear answers.

I. (5) Use a double integral in polar coordinates to calculate the area bounded by the circle $x^2 + y^2 = ax$.

In polar coordinates the equation is $r^2 = ar\cos(\theta)$, so the circle is described by $r = a\cos(\theta)$ for $-\pi/2 \le \theta \le \pi/2$. Writing R for the domain, we have $\iint_R dR = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos(\theta)} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} a^2\cos^2(\theta)/2d\theta = \int_{-\pi/2}^{\pi/2} a^2(1+\cos(2\theta))/4\,d\theta = a^2\theta/4 + \sin(2\theta)/8\Big|_{-\pi/2}^{\pi/2} = \pi a^2/4$.

II. (4) Rewrite the integral $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{0} f(x,y)\,dy\,dx$ to integrate first with respect to x, then with respect to y.

Since y goes from $-\sqrt{3-x^2}$ to 0 for each x between $-\sqrt{3}$ and $\sqrt{3}$, the domain is the bottom half of the standard disk of radius $\sqrt{3}$. Integrating first with respect to x gives $\displaystyle\int_{-\sqrt{3}}^{0}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} f(x,y)\,dx\,dy$.

III. (6) Consider the solid cylinder bounded by $x^2 + z^2 = 4$ and the planes $y = 0$ and $y = 1$, and let E be the points in this solid cylinder that have $z \ge 0$. Suppose that the density of E at a point (x, y, z) equals twice the distance from (x, y, z) to the y-axis. Calculate the mass of E.

For each (x, z) in T, the upper half disk of radius 2 in the xz-plane, y goes from 0 to 1. The distance from a point (x, y, z) to the y axis is $\sqrt{x^2+z^2}$, so $\rho = 2\sqrt{x^2+z^2}$. Therefore the mass is given by $\displaystyle\iiint_E \rho\,dV = \iint_R\int_0^2 2\sqrt{x^2+z^2}\,dy\,dR = \iint_R 2\sqrt{x^2+z^2}\,dR = \int_0^{\pi}\int_0^2 2r\,r\,dr\,d\theta = \frac{16\pi}{3}$.

IV. (5) Let D be the unit disk in the xy-plane. Write an integral in polar coordinates to calculate the surface area of the portion of $z = e^{x^2+y^2}$ lying above D. Supply limits of integration, but do not continue further in evaluation of the integral.

First we calculate

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \left(2xe^{x^2+y^2}\right)^2 + \left(2ye^{x^2+y^2}\right)^2} = \sqrt{1 + 4(x^2+y^2)e^{2(x^2+y^2)}}$$

so the surface area is given by $\displaystyle\iint_D \sqrt{1+4(x^2+y^2)e^{2(x^2+y^2)}}\,dD = \int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2e^{2r^2}}\,dr\,d\theta$.

V. (4) Let D be the unit disk in the xy-plane, and consider the function $f(x,y) = \dfrac{1}{e^{x^2+y^2}}$, whose values depend only on r. Obtain a partition of D by cutting it into four quarters, each consisting of the intersection of D with one of the four quadrants. Calculate the smallest and largest Riemann sums for f for this partition.

In polar coordinates, the function is $f(r,\theta) = e^{-r^2}$, so on D its largest is 1, when $r = 0$, and its smallest values when r is $1/e$, when $r = 1$. These are also the extreme values on each quarter disk, and each quarter disk has area $\pi/4$. So the smallest Riemann sum is $\sum_{i=1}^4 (1/e)\cdot\pi/4 = \pi/e$, and the largest is $\sum_{i=1}^4 1\cdot\pi/4 = \pi$.

VI. (8) Calculate the area inside the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ as follows.

(a) Let S be the region bounded by the ellipse. Define ϕ from the uv-plane to the xy-plane by $\phi(u,v) = (au, bv)$. Determine the region R in the uv-plane that corresponds to S under ϕ.

In (u,v)-coordinates, the boundary ellipse becomes $\dfrac{(au)^2}{a^2} + \dfrac{(bv)^2}{b^2} = 1$, that is, $u^2 + v^2 = 1$, so R is the unit disk.

(b) Calculate the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{pmatrix}$ and its determinant.

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, whose determinant is ab.

(c) Write a double integral over the domain S whose value is the area, change it into uv-coordinates, and evaluate to find the area.

$$\iint_S dS = \iint_R ab\, dR = ab \iint_R dR = ab \cdot (\text{area of } R) = \pi ab.$$

VII. (6) Consider the function $f(x,y) = x^4 + y^4 - 4xy + 2$ on the square $D = \{(x,y) \mid -2 \le x \le 2, -2 \le y \le 2$.

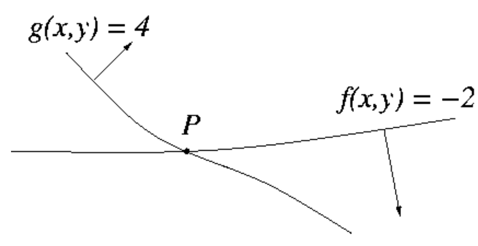
(a) Find all critical points of f on this domain.

We have $\dfrac{\partial f}{\partial x} = 4x(x^2 - 1)$ and $\dfrac{\partial f}{\partial y} = 4y(y^2 - 1)$. The first is 0 exactly when x is -1, 0, or 1, and the second is 0 exactly when y is -1, 0, or 1, so the nine critical points are $(-1,-1)$, $(-1,0)$, $(-1,1)$, $(0,-1)$, $(0,0)$, $(0,1)$, $(1,-1)$, $(1,0)$, and $(1,1)$, all of which lie in D.

(b) Show that an extreme value of f on the right-hand vertical boundary side of D can occur only at one of the three points $(2, \pm 2)$ or $(2, \sqrt[3]{2})$.

On the right side, $x = 2$, so the function becomes $f(2,y) = y^4 - 8y + 18$ for $-2 \le y \le 2$. Its derivative with respect to y is $4y^3 - 8$, so its only critical point is when $y = \sqrt[3]{2})$. Any extreme values must occur either at this point, or at an endpoint ± 2, giving the three possibilites $(2, \pm 2)$ and $(2, \sqrt[3]{2})$.

VIII. (4) The figure to the right shows level curves $f(x,y) = -2$ and $g(x,y) = 4$ of two functions f and g in the xy-plane, their intersection point P, and a gradient vector for each of the functions at a point on its level curve. Let h be the function defined by $h(x,y) = f(x,y)\, g(x,y)$. There are four directions in which one can leave P moving along one of the level curves. For each of them, label whether h is increasing or decreasing as one moves away from P in that direction. (Caution: Before answering, think carefully about the way that a product changes when one of the factors is negative.)



Suppose, for example, that one moves east along the level curve of f. The value of f remains at -2, while the value of g increases from 4, since the gradient of g indicates that g increases as we move east from its level curve. Since the value 4 of g *increases*, the value $4 \cdot (-2)$ of h *decreases*. Similarly, moving west along that level curve increases h, moving northwest along the level curve of g decreases h, and moving southeast along the level curve of g increases h.

IX. Use integration in spherical coordinates to find the volume of the region E bounded by $z = \sqrt{x^2 + y^2}$ and
(6) $x^2 + y^2 + z^2 = 3$.

ρ goes from 0 to the radius $\sqrt{3}$, ϕ from 0 on the z-axis to $\pi/4$ on the cone $z = \sqrt{x^2+y^2}$, and θ from 0 to 2π. So we have

$$V = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{3}} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = 2\pi \cdot \frac{(\sqrt{3})^3}{3} \cdot (1 - \frac{1}{\sqrt{2}}) = \pi\sqrt{3}(2 - \sqrt{2}) \ .$$

X. Let E be the upper hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, and consider the integral $\iiint_E f(x, y, z)\, dV$.
(6) (a) Rewrite the integral in cylindrical coordinates, supplying limits of integration appropriate for E.

In cylindrical coordinates, the hemisphere is $r^2 + z^2 = a^2$, or $z = \sqrt{a^2 - r^2}$, so the integral becomes $\displaystyle\int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} f(r\cos(\theta), r\sin(\theta), z)\, r\, dz\, dr\, d\theta$.

(b) Rewrite the integral in spherical coordinates, supplying limits of integration appropriate for E.

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^a f(\rho\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$