Mathematics 2443-006H Name (please print) ____________________

Examination I

February 14, 2006

Instructions: Give brief, clear answers. All functions are assumed to have continuous derivatives of all orders, so results such as Clairaut's Theorem may be freely applied if needed.

I. (6) Let $f(x,y)$ be a function of two variables, and let (x_0, y_0) be a point in the xy-plane. Consider the curve given by the vector-valued function $\vec{r}(t) = (x_0+t)\vec{\imath} + y_0\vec{\jmath} + f(x_0+t, y_0)\vec{k}$.

1. Calculate $\vec{r}\,'(t)$ (to find $\dfrac{df(x_0+t,y_0)}{dt}$, you may need the Chain Rule). Calculate $\vec{r}\,'(0)$.

$$\vec{r}\,'(t) = 1\vec{\imath} + 0\vec{\jmath} + \frac{df(x_0+t,y_0)}{dt}\vec{k} = \vec{\imath} + \left(f_x(x_0+t,y_0)\cdot\frac{d(x_0+t)}{dt} + f_y(x_0+t,y_0)\cdot\frac{d(y_0)}{dt}\right)\vec{k}$$
$$= \vec{\imath} + (f_x(x_0+t,y_0)\cdot 1 + f_y(x_0+t,y_0)\cdot 0)\vec{k} = \vec{\imath} + f_x(x_0+t,y_0)\vec{k}.$$

so $\vec{r}\,'(0) = \vec{\imath} + f_x(x_0,y_0)\vec{k}$.

2. Draw a sketch of the graph of f, the curve, and $\vec{r}\,'(0)$.

(see last page)

II. (12) Consider the helix $x = \cos(t)$, $y = \sin(t)$, and $z = ct$ where c is a positive constant. Its velocity vector is $\vec{v}(t) = -\sin(t)\vec{\imath} + \cos(t)\vec{\jmath} + c\vec{k}$, so its speed $\dfrac{ds}{dt}$ is $\|\vec{v}(t)\| = \sqrt{1+c^2}$.

1. Calculate the unit tangent vector $\vec{T}$.

$$\vec{T} = \vec{v}/\|\vec{v}\| = (-\sin(t)\vec{\imath} + \cos(t)\vec{\jmath} + c\vec{k})/\sqrt{1+c^2} = -\frac{\sin(t)}{\sqrt{1+c^2}}\vec{\imath} + \frac{\cos(t)}{\sqrt{1+c^2}}\vec{\jmath} + \frac{c}{\sqrt{1+c^2}}\vec{k}.$$

2. Use the Chain Rule $\dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{ds}\dfrac{ds}{dt}$ to calculate $\dfrac{d\vec{T}}{ds}$ and the curvature $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\|$.

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\Big/\frac{ds}{dt} = \left(\frac{1}{\sqrt{1+c^2}}(-\cos(t)\vec{\imath} - \sin(t)\vec{\jmath})\right)\Big/\sqrt{1+c^2} = \frac{1}{1+c^2}(-\cos(t)\vec{\imath} - \sin(t)\vec{\jmath}),$$
so $\kappa = \dfrac{1}{1+c^2}\| -\cos(t)\vec{\imath} - \sin(t)\vec{\jmath}\| = \dfrac{1}{1+c^2}$.

3. Calculate the unit normal $\vec{N} = \vec{T}'/\|\vec{T}'\|$, and verify that $\dfrac{d\vec{T}}{ds} = \kappa\vec{N}$.

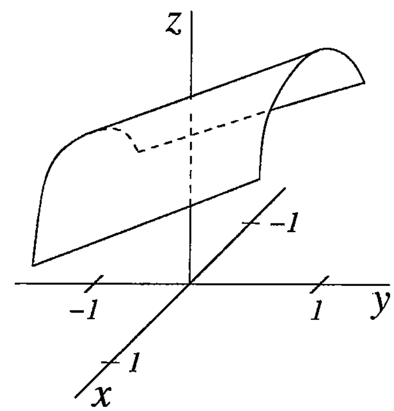
From part (b), $\vec{T}' = \dfrac{1}{\sqrt{1+c^2}}(-\cos(t)\vec{\imath} - \sin(t)\vec{\jmath})$, so
$\|\vec{T}'\| = \dfrac{1}{\sqrt{1+c^2}}\| -\cos(t)\vec{\imath} - \sin(t)\vec{\jmath}\| = \dfrac{1}{\sqrt{1+c^2}}$ and $\vec{N} = -\cos(t)\vec{\imath} - \sin(t)\vec{\jmath}$.

4. Calculate the binormal $\vec{B} = \vec{T}\times\vec{N}$, and use the formula $\dfrac{d\vec{B}}{ds} = -\tau\vec{N}$ to calculate the torsion τ.

$$\vec{B} = \vec{T}\times\vec{N} = \frac{c}{\sqrt{1+c^2}}\sin(t)\vec{\imath} - \frac{c}{\sqrt{1+c^2}}\cos(t)\vec{\jmath} + \frac{1}{\sqrt{1+c^2}}\vec{k},$$
so $\dfrac{d\vec{B}}{ds} = \dfrac{d\vec{B}}{dt}\Big/\dfrac{ds}{dt} = \left(\dfrac{c}{\sqrt{1+c^2}}\cos(t)\vec{\imath} + \dfrac{c}{\sqrt{1+c^2}}\sin(t)\vec{\jmath}\right)\Big/\sqrt{1+c^2}$
$= \dfrac{c}{1+c^2}(\cos(t)\vec{\imath} + \sin(t)\vec{\jmath}) = -\dfrac{c}{1+c^2}\vec{N}$, giving $\tau = \dfrac{c}{1+c^2}$. (Of course, when $c = 0$ the curve is planar so $\tau = 0$, but more interestingly, the maximum torsion occurs when $c = 1$, and the torsion limits to 0 as $c \to \infty$. Does this make sense to you geometrically?)

III. (4) In an xy-coordinate system, sketch the gradient of the function whose graph is shown to the right.

(see last page)



IV. (4) Use implicit differentiation to calculate $\left.\dfrac{\partial R}{\partial R_3}\right|_{(R_1,R_2,R_3)=(\sqrt{3},\sqrt{6},2)}$ if $\dfrac{1}{R} = \dfrac{1}{R_1^2} + \dfrac{1}{R_2^2} + \dfrac{1}{R_3^2}$.

$-\dfrac{1}{R^2}\dfrac{\partial R}{\partial R_3} = 0 + 0 - \dfrac{2}{R_3^3}$, so $\dfrac{\partial R}{\partial R_3} = \dfrac{2R^2}{R_3^3}$. When $(R_1, R_2, R_3) = (\sqrt{3}, \sqrt{6}, 2)$, $R = 4/3$, so $\left.\dfrac{\partial R}{\partial R_3}\right|_{(R_1,R_2,R_3)=(\sqrt{3},\sqrt{6},2)} = \dfrac{2(\frac{4}{3})^2}{8} = \dfrac{4}{9}$.

V. (4) Five positive numbers x, y, z, u, and v, each less than or equal to 10, are multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from rounding each number off to the nearest whole number.

We calculate $d(xyzuv) = yzuv\,dx + xzuv\,dy + xyuv\,dz + xyzv\,du + xyzu\,dv$. Rounding off to the nearest integer allows any of dx, etc., to be as large as 0.5, and each of the four-term products is at most 10,000, so the linear part of the error is no more than $5 \cdot 10{,}000 \cdot 0.5 = 25{,}000$.

VI. (4) Five positive numbers x, y, z, u, and v, each less than or equal to 10, are multiplied together. The first three are increasing at 2 units per second, while the last two are decreasing at 4 units per second. Find the rate of change of the product at a moment when each of the numbers equals 10.

The Chain Rule gives

$$\frac{d(xyzuv)}{dt} = yzuv\frac{dx}{dt} + xzuv\frac{dy}{dt} + xyuv\frac{dz}{dt} + xyzv\frac{du}{dt} + xyzu\frac{dv}{dt}\ .$$

Specializing to a moment when each of x, etc. equals 10, we obtain

$$10{,}000 \cdot 2 + 10{,}000 \cdot 2 + 10{,}000 \cdot 2 - 10{,}000 \cdot 4 - 10{,}000 \cdot 4 = -20{,}000\ .$$

VII. (6) If z is a function of x and y, calculate $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$, where r and θ are the polar coordinates. Write each result in terms of $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, x, y, and r, that is, without using θ explicitly.

We have $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so using the Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial(r\cos(\theta))}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial(r\sin(\theta))}{\partial r} = \cos(\theta)\frac{\partial z}{\partial x} + \sin(\theta)\frac{\partial z}{\partial y} = \frac{1}{r}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial(r\cos(\theta))}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial(r\sin(\theta))}{\partial \theta} = -r\sin(\theta)\frac{\partial z}{\partial x} + r\cos(\theta)\frac{\partial z}{\partial y} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

VIII. Calculate each of the following.
(12) (a) The directional derivative of $\dfrac{1}{xy}+\dfrac{1}{yz}$ at $(2,1,2)$ in the direction toward the origin.

We have $\nabla\left(\dfrac{1}{xy}+\dfrac{1}{yz}\right)=-\dfrac{1}{x^2y}\vec{\imath}-\left(\dfrac{1}{xy^2}+\dfrac{1}{y^2z}\right)\vec{\jmath}-\dfrac{1}{yz^2}\vec{k}$. so the gradient at $(2,1,2)$ is $-\frac{1}{4}\vec{\imath}-\vec{\jmath}-\frac{1}{4}\vec{k}$. A vector in the direction of the origin is $-2\vec{\imath}-j-2\vec{k}$, whose length is 3, so a unit vector in the direction of the origin is $-\frac{2}{3}\vec{\imath}-\frac{1}{3}j-\frac{2}{3}\vec{k}$. Taking the dot product of the gradient vector with this unit vector gives $\frac{2}{3}$.

(b) The maximum rate of change of $qe^{-p}-pe^{-q}$ at $(p,q)=(0,0)$, and the direction in which it occurs.

We calculate $\nabla(qe^{-p}-pe^{-q})=(-qe^{-p}-e^{-q})\vec{\imath}+(e^{-p}+pe^{-q})\vec{\jmath}$. At this origin, this is $-\vec{\imath}+\vec{\jmath}$, so this is the direction of the maximum rate of change, and this maximum is $\|-\vec{\imath}+\vec{\jmath}\|=\sqrt{2}$.

(c) A vector-valued function giving the line perpendicular to the level surface of xyz at the point $(1,2,3)$.

We calculate $\nabla(xyz)=yz\vec{\imath}+xz\vec{\jmath}+xy\vec{k}$, whose value at $(1,2,3)$ is $6\vec{\imath}+3\vec{\jmath}+2\vec{k}$. This is a direction vector for the normal line, which is then given by the vector-valued function $\vec{r}(t)=(1+6t)\vec{\imath}+(2+3t)\vec{\jmath}+(3+2t)\vec{k}$.

(d) An equation for the tangent plane to the level surface of $\dfrac{1}{xyz}$ at the point $(1,2,3)$.

The level surfaces of $\dfrac{1}{xyz}$ are the same as those of the function xyz. We already calculated the gradient vector of xyz at this point to be $6\vec{\imath}+3\vec{\jmath}+2\vec{k}$, and it is a normal vector to the tangent plane. So an equation for the tangent plane is $6(x-1)+3(y-2)+2(z-3)=0$, or $6x+3y+2z=18$.

IX. Suppose that z is a function of x and y for which $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=0$. Show that $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2 z}{\partial y^2}$.
(4)

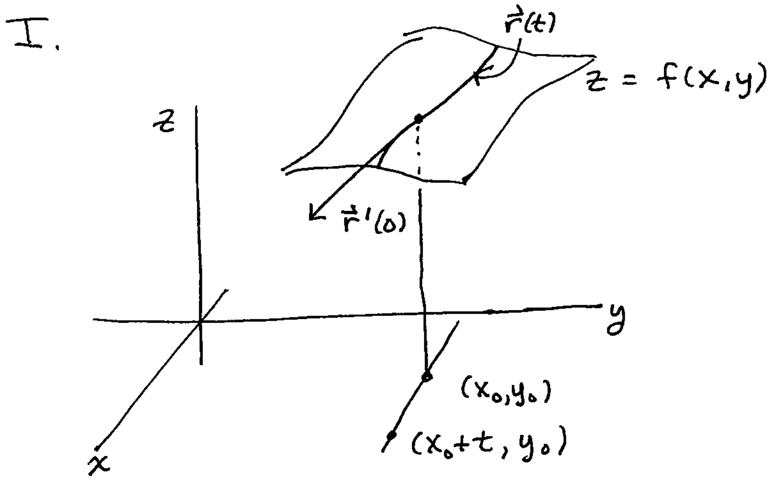
Applying $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ to the given equation, we obtain

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\,\partial y}=0$$
$$\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial^2 z}{\partial y^2}=0$$

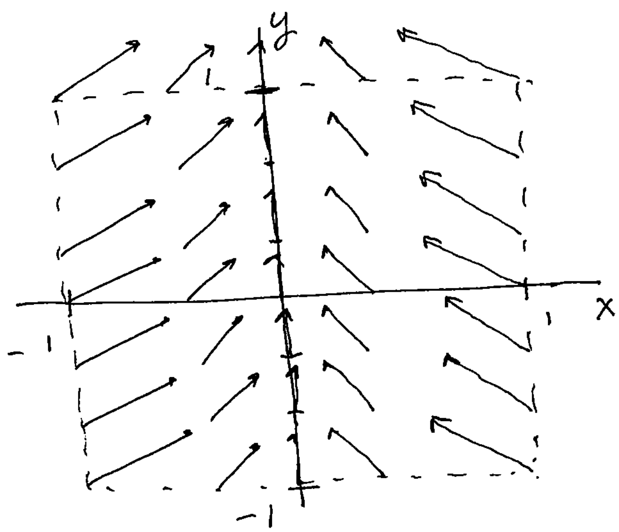
Using Clairaut's Theorem, we have

$$\frac{\partial^2 z}{\partial x^2}=-\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial^2 z}{\partial y^2}\,.$$

I.



III.



Note: All gradient vectors have the same $\vec{j}$ component, since $\frac{\partial f}{\partial y}$ is constant.