Mathematics 2443-006H Name (please print) ________________

Examination I

February 14, 2006

Instructions: Give brief, clear answers. All functions are assumed to have continuous derivatives of all orders, so results such as Clairaut's Theorem may be freely applied if needed.

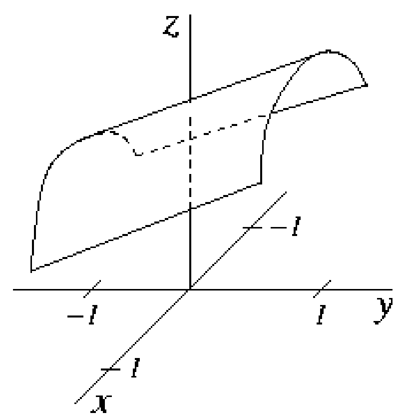
I. (6) Let $f(x,y)$ be a function of two variables, and let (x_0, y_0) be a point in the xy-plane. Consider the curve given by the vector-valued function $\vec{r}(t) = (x_0+t)\vec{\imath} + y_0\vec{\jmath} + f(x_0+t, y_0)\vec{k}$.

1. Calculate $\vec{r}\,'(t)$ (to find $\dfrac{df(x_0+t, y_0)}{dt}$, you may need the Chain Rule). Calculate $\vec{r}\,'(0)$.
2. Draw a sketch of the graph of f, the curve, and $\vec{r}\,'(0)$.

II. (12) Consider the helix $x = \cos(t)$, $y = \sin(t)$, and $z = ct$ where c is a positive constant. Its velocity vector is $\vec{v}(t) = -\sin(t)\vec{\imath} + \cos(t)\vec{\jmath} + c\vec{k}$, so its speed $\dfrac{ds}{dt}$ is $\|\,\vec{v}(t)\,\| = \sqrt{1+c^2}$.

1. Calculate the unit tangent vector $\vec{T}$.
2. Use the Chain Rule $\dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{ds}\,\dfrac{ds}{dt}$ to calculate $\dfrac{d\vec{T}}{ds}$ and the curvature $\kappa = \left\|\dfrac{d\vec{T}}{ds}\right\|$.
3. Calculate the unit normal $\vec{N} = \vec{T}\,'/\|\,\vec{T}\,'\,\|$, and verify that $\dfrac{d\vec{T}}{ds} = \kappa\,\vec{N}$.
4. Calculate the binormal $\vec{B} = \vec{T}\times\vec{N}$, and use the formula $\dfrac{d\vec{B}}{ds} = -\tau\vec{N}$ to calculate the torsion τ.

III. (4) In an xy-coordinate system, sketch the gradient of the function whose graph is shown to the right.



IV. (4) Use implicit differentiation to calculate $\left.\dfrac{\partial R}{\partial R_3}\right|_{(R_1,R_2,R_3)=(\sqrt{3},\sqrt{6},2)}$ if $\dfrac{1}{R} = \dfrac{1}{R_1^2} + \dfrac{1}{R_2^2} + \dfrac{1}{R_3^2}$.

V. (4) Five positive numbers x, y, z, u, and v, each less than or equal to 10, are multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from rounding each number off to the nearest whole number.

VI. (4) Five positive numbers x, y, z, u, and v, each less than or equal to 10, are multiplied together. The first three are increasing at 2 units per second, while the last two are decreasing at 4 units per second. Find the rate of change of the product at a moment when each of the numbers equals 10.

VII. (6) If z is a function of x and y, calculate $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$, where r and θ are the polar coordinates. Write each result in terms of $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$, x, y, and r, that is, without using θ explicitly.

VIII. (12) Calculate each of the following.

(a) The directional derivative of $\dfrac{1}{xy} + \dfrac{1}{yz}$ at $(2, 1, 2)$ in the direction toward the origin.

(b) The maximum rate of change of $qe^{-p} - pe^{-q}$ at $(p, q) = (0, 0)$, and the direction in which it occurs.

(c) A vector-valued function giving the line perpendicular to the level surface of xyz at the point $(1, 2, 3)$.

(d) An equation for the tangent plane to the level surface of $\dfrac{1}{xyz}$ at the point $(1, 2, 3)$.

IX. (4) Suppose that z is a function of x and y for which $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$. Show that $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$.