Mathematics 5863 Name (please print) ____________________

Examination II

April 7, 2005

Instructions: Give brief, clear answers, avoiding excessive details (especially when the instruction is "Verify"). For example, if $F\colon \alpha \simeq_p \beta$ is a path homotopy, you need not verify that $f \circ F$ is a path homotopy from $f \circ \alpha$ to $f \circ \beta$. If you lift a path homotopy, you need not verify that the lifted homotopy is a path homotopy.

Even in problems where you cannot do one of the early parts of the problems, try to solve later parts of the problem taking the previous parts as known.

I. (15) Let X be a path-connected space. In this question, you may assume the facts that if each $\alpha_i \simeq_p \alpha_i'$, then $\alpha_1 * \cdots * \alpha_n \simeq_p \alpha_1' * \cdots * \alpha_n'$, and that $c_{\alpha(0)} * \alpha \simeq_p \alpha \simeq_p \alpha * c_{\alpha(1)}$.

(a) Suppose that γ is a path in X. Define the change-of-basepoint function $h_\gamma\colon \pi_1(X, \gamma(1)) \to \pi_1(X, \gamma(0))$.

$h_\gamma(\langle\alpha\rangle) = \langle \gamma * \alpha * \overline{\gamma}\rangle$.

(b) Verify that if τ is another path in X, with $\tau(0) = \gamma(1)$, then $h_{\gamma*\tau} = h_\gamma \circ h_\tau$.

$h_{\gamma*\tau}(\langle\alpha\rangle) = \langle \gamma * \tau * \alpha * \overline{\gamma * \tau}\rangle = \langle \gamma * \tau * \alpha * \overline{\tau} * \overline{\gamma}\rangle = h_\gamma(\langle \tau * \alpha * \overline{\tau}\rangle) = h_\gamma(h_\tau(\langle\alpha\rangle)) = h_\gamma \circ h_\tau(\langle\alpha\rangle)$.

(c) Verify that if $\gamma \simeq_p \gamma'$, $h_\gamma = h_{\gamma'}$.

$h_\gamma(\langle\alpha\rangle) = \langle \gamma * \alpha * \overline{\gamma}\rangle = \langle \gamma' * \alpha * \overline{\gamma'}\rangle = h_{\gamma'}(\langle\alpha\rangle)$.

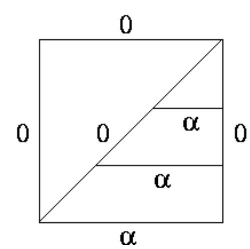
(d) Verify that $h_{c_{x_0}} = id_{\pi_1(X,x_0)}$, where as usual c_{x_0} denotes the constant path at a point $x_0 \in X$.

$h_{c_{x_0}}(\langle\alpha\rangle) = \langle c_{x_0} * \alpha * \overline{c_{x_0}}\rangle = \langle c_{x_0} * \alpha * c_{x_0}\rangle = \langle \alpha * c_{x_0}\rangle = \langle\alpha\rangle$.

(e) From the previous facts, deduce that h_γ is bijective.

$h_{\overline{\gamma}} \circ h_\gamma = h_{\overline{\gamma}*\gamma} = h_{c_{\gamma(0)}} = id_{\pi_1(X,\gamma(0))}$ and consequently $h_\gamma \circ h_{\overline{\gamma}} = h_{\overline{\overline{\gamma}}} \circ h_{\overline{\gamma}} = id_{\pi_1(X,\gamma(1))}$, showing that h_γ is injective and surjective respectively.

II. (5) Let $\alpha\colon I \to \mathbb{R}$ be the loop at 0 defined by $\alpha(t) = \sin(\pi t)$. Show that the function $F\colon I \times I \to \mathbb{R}$ indicated by the diagram shown at the right is not continuous.



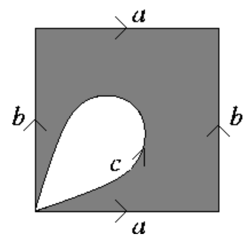
Consider $C = F^{-1}([1/2, \infty))$. If F were continuous, then this would be a closed set. But for $s < 1$, C meets each the horizontal subinterval in $I \times \{s\}$ indicated by the ones labeled α in the figure, while C is disjoint form the top of the square $I \times \{1\}$. So C cannot be closed.

III. (5) Let $f\colon (X, x_0) \to (Y, y_0)$ be a map taking basepoint to basepoint. Show that the induced homomorphism $f_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $f_\#(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$ is well-defined.

Suppose that $\langle\alpha\rangle = \langle\alpha'\rangle$, and let $F\colon \alpha \simeq_p \alpha'$. Then $f \circ F\colon f \circ \alpha \simeq_p f \circ \alpha'$, so $\langle f \circ \alpha\rangle = \langle f \circ \alpha'\rangle$.

IV. (10) Let $F = T\#D$ be the orientable surface of genus 1 with one boundary circle, let $x_0 \in \partial F$, and let $i\colon \partial F \to F$ denote the inclusion. We know that $\pi_1(\partial F, x_0) \cong \mathbb{Z}$, since $\partial F \approx S^1$.

(a) Give an explanation (the right picture would be very helpful) of why $\pi_1(F, x_0)$ contains two elements a and b for which $aba^{-1}b^{-1} = i_\#(c)$ where c generates $\pi_1(\partial F, x_0)$.



(b) Use the fact in (a) to prove that ∂F is not a retract of F.

Suppose for contradiction that there exists a retraction $r\colon F \to \partial F$, so that $r \circ i = id_{\partial F}$. Using the functorial properties, we have $r_\# \circ i_\# = id_{\pi_1(\partial F, x_0)}$. But then, $c = r_\# \circ i_\#(c) = r_\#(aba^{-1}b^{-1}) = r_\#(a)r_\#(b)r_\#(a)^{-1}r_\#(b)^{-1} = r_\#(a)r_\#(a)^{-1}r_\#(b)r_\#(b)^{-1} = 1$, using the fact that $\pi_1(\partial F, x_0)$ is abelian. This contradicts the fact that c generates $\pi_1(\partial F, x_0)$.

V. (15) Recall that a simply-connected space is path-connected and has the property that any two paths with the same starting and ending points are path homotopic. Let X be a simply-connected space with basepoint x_0, let $f\colon X \to S^1$ be any continuous map, and suppose that $r_0 \in \mathbb{R}$ with $p(r_0) = f(x_0)$. Define $\widetilde{f}\colon X \to \mathbb{R}$ as follows: For each $x \in X$, choose a path $\alpha\colon I \to X$ from x_0 to x, let $\widetilde{f \circ \alpha}\colon I \to \mathbb{R}$ be the unique lift of $f \circ \alpha$ starting at r_0, and define $\widetilde{f}(x) = \widetilde{f \circ \alpha}(1)$.

(a) Verify that $\widetilde{f}$ is well-defined.

Suppose that β is any other path from x_0 to x. Since X is simply-connected, there exists a path homotopy $F\colon \alpha \simeq_p \beta$, and $f \circ F\colon I \times I \to S^1$ is also a path homotopy, and $f \circ F(0,0) = f(x_0)$. Let $\widetilde{F}\colon I \times I \to \mathbb{R}$ be the uniuqe lift of $f \circ F$ taking $(0,0)$ to r_0. It is a path homotopy from $\widetilde{f \circ \alpha}$ to $\widetilde{f \circ \beta}$, so $\widetilde{f \circ \alpha}(1) = \widetilde{f \circ \beta}(1)$.

(b) Verify that $p \circ \widetilde{f} = f$.

$p \circ \widetilde{f}(x) = p(\widetilde{f \circ \alpha}(1)) = f \circ \alpha(1) = f(x)$.

(c) Take as known the fact that $\widetilde{f}$ is continuous, that is, that $\widetilde{f}$ is a lift of f. Prove that if $n \geq 2$ then any continuous map $f\colon S^n \to S^1$ is homotopic to a constant map.

Let $\widetilde{f}\colon X \to \mathbb{R}$ be the lift of f that we have constructed. Since $\mathbb{R}$ is convex, any map from X to $\mathbb{R}$ is homotopic to a constant map. So $f = p \circ \widetilde{f} \simeq p \circ c$ for some constant map c, and $p \circ c$ is constant.

VI. (10) Let $p\colon \mathbb{R} \to S^1$ be the usual covering map $p(r) = e^{2\pi i r}$. Let $s_0 = (1,0) = p(0) \in S^1$. Define $\Phi\colon \pi_1(S^1, s_0) \to \mathbb{Z}$ by $\Phi(\langle\alpha\rangle) = \widetilde{\alpha}(1)$, where $\widetilde{\alpha}$ is the lift of α to $\mathbb{R}$ starting at 0 (take as known the fact that Φ is well-defined).

(a) Prove that Φ is injective.

Suppose that $\Phi(\langle\alpha\rangle) = \Phi(\langle\beta\rangle)$. Then $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$. Since also $\widetilde{\alpha}(0) = \widetilde{\beta}(0)$, and $\mathbb{R}$ is simply-connected, $\widetilde{\alpha} \simeq_p \widetilde{\beta}$. Let $F\colon \widetilde{\alpha} \simeq_p \widetilde{\beta}$. Then $p \circ F\colon \alpha \simeq_p \beta$, so $\langle\alpha\rangle = \langle\beta\rangle$.

(b) Prove that Φ is surjective.

Let $n \in \mathbb{Z}$ and let $\gamma_n\colon I \to \mathbb{R}$ be the path defined by $\gamma_n(t) = nt$. Since $\gamma_n(1) = n$, $p \circ \gamma_n$ is a loop in S^1 based at $(1,0)$, and γ_n is its lift beginning at 0. Therefore $\Phi(\langle p \circ \gamma_n\rangle = \gamma_n(1) = n$.

VII. Let $f\colon S^1 \to S^1$ be a continuous map.
(15)

(a) Define the *degree* of f.

Let $\alpha\colon I \to S^1$ be the loop defined by $\alpha(t) = e^{2\pi i t}$. Let $\widetilde{f \circ \alpha}\colon I \to \mathbb{R}$ be any lift of the path $f \circ \alpha$. Then $\deg(f) = \widetilde{f \circ \alpha}(1) - \widetilde{f \circ \alpha}(0)$.

(b) Verify that the map $f_n\colon S^1 \to S^1$ defined by $f_n(z) = z^n$ has degree n.

Let $\gamma\colon I \to \mathbb{R}$ be $\gamma(t) = nt$. Then $p \circ \gamma(t) = e^{2\pi i n t} = (e^{2\pi i t})^n = f_n \circ \alpha(t)$, and $\gamma(0) = 0$, so $\gamma = \widetilde{f_n \circ \alpha}$. Therefore the degree of f_n is $\gamma(1) - \gamma(0) = n$.

(c) Taking as known the fact that the degree of a composition is the product of degrees, verify that if f is a homeomorphism, the f has degree 1 or -1.

Since $id_{S^1} = f_1$, part (b) shows that $\deg(id_{S^1}) = 1$. Therefore $1 = \deg(f^{-1} \circ f) = \deg(f^{-1})\deg(f)$. That is, $\deg(f)$ divides 1, so $\deg(f) = \pm 1$.

(d) Taking as known the fact that two maps from S^1 to S^1 are homotopic to if and only if they have the same degree, prove that if $f\colon S^1 \to S^1$ is homotopic to a homeomorphism, then f is homotopic to exactly one of the maps id_{S^1} or f_{-1}.

From part (b), these maps have degrees 1 and -1 respectively. By (c), any homeomorphism has degree 1 or -1, so is homotopic to either id_{S^1} or f_{-1}. It cannot be homotopic to both, because homotopic maps have the same degree.