Mathematics 5863 Name (please print) ____________________

Examination I

February 22, 2005

Instructions: Give brief, clear answers.

I. (6) Let M be a manifold with boundary. What is a *collar* of ∂M? Draw a picture of a Möbius band, showing a collar on its boundary.

A collar of ∂M is an imbedding $\partial M \times I \to M$ that sends $(x, 0)$ to x for all $x \in \partial M$ (or, that sends $(x, 1)$ to x for all $x \in \partial M$). For the picture see the last page.

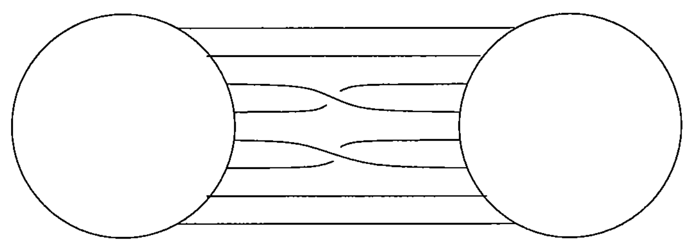
II. (5) Let X be obtained from a disk by attaching two untwisted 1-handles whose ends alternate in the boundary of the disk. Draw a picture of X imbedded in a torus.

See the last page.

III. (10) Prove that every contractible space is path-connected.

Suppose that X is contractible, and let $F\colon id_X \simeq c_{x_0}$ where $c_{x_0}\colon X \to X$ is the constant map that sends every point x to x_0. For any $x \in X$, define $\alpha\colon I \to X$ by $\alpha(t) = F(x, t)$. This is path from x to x_0 (it is continuous since it is the restriction of F to $\{x\} \times I$ preceded by the inclusion $I \to X \times I$ that sends t to (x, t)), so every point of X is in the path component of x_0.

IV. (20) A compact connected 2-manifold is shown at the right. It has a handle structure with two 0-handles and four 1-handles.



1. Use Euler characteristic and orientability to determine the homeomorphism type of this 2-manifold.
2. Determine the homeomorphism type directly by using handle slides to simplify this handle structure into a standard form.

We observe that the surface is nonorientable, and has two boundary circles. It Euler characteristic is $2 - 4 = -2$, so we have $2 - n - 2 = -2$ and therefore $n = 2$. So the surface is $P\#P\#D\#D$. For the handle sliding, see the last page.

V. (5) Let $f\colon S^1 \to S^1$ be the homeomorphism sending θ to $\theta + \pi$. Construct an explicit isotopy from id_{S^1} to f.

Define $F\colon S^1 \times I \to S^1$ by $F(\theta, t) = \theta + t\pi$. Each F_t is a homeomorphism (with inverse sending θ to $\theta - t\pi$), $F_0 = id_{S^1}$, and $F_1(\theta) = \theta + \pi$.

VI. (10) Let $\gamma\colon I \to S^1$ be the path defined by $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$. Verify that $\gamma * (\gamma * \gamma) \neq (\gamma * \gamma) * \gamma$.

$\gamma*(\gamma*\gamma)(1/4) = \gamma(2\cdot 1/4) = \gamma(1/2) = (-1, 0)$, but $(\gamma*\gamma)*\gamma(1/4) = \gamma*\gamma(2\cdot 1/4) = \gamma(2\cdot 2\cdot 1/4) = \gamma(1) = (1, 0)$.

VII. (5) The following is an incorrect proof of the true fact that $\pi_1(S^2, x_0)$ is the trivial group. Find the error in it: Let $\alpha\colon I \to S^2$ be a loop based at x_0. Choose a point $x_1 \in S^2$ which is not in the image of α. Since $S^2 - \{x_1\}$ is homeomorphic to $\mathbb{R}^2$, and any two loops based at the same point in $\mathbb{R}^2$ are path-homotopic, α is path-homotopic to the constant path at x_0. Therefore $\pi_1(S^2, x_0)$ is the trivial group.

If α is surjective, then a point x_1 not in the image of α does not exist.

VIII. (10) Use the facts that $\pi_1(S^1, s_0) \cong \mathbb{Z}$ and $\pi_1(D^2, s_0) \cong \{0\}$, together with the functorial properties of the induced homomorphism, to prove that the circle is not a retract of the disk.

Let $i\colon S^1 \to D^2$ be the inclusion and suppose for contradiction that there exists a retraction $r\colon D^2 \to S^1$, so that $r \circ i = id_{S^1}$. Then $id_{\pi_1(S^1,s_0)} = (id_{S^1})_\# = (r \circ i)_\# = r_\# \circ i_\#\colon \pi_1(S^1, s_0) \to \pi_1(D^2, s_0) \to \pi_1(S^1, s_0)$, but this would be a sequence of homomorphisms $\mathbb{Z} \to \{0\} \to \mathbb{Z}$ whose composition is the identity.

IX. (20) Prove or give a counterexample:

1. A connected sum $M\#M$ can be homeomorphic to M.

   True, for example $S^2\#S^2 = S^2$.

2. The property of being contractible is a topological invariant.

   Yes, if $h\colon X \to Y$ is a homeomorphism, and X is contractible with $id_X \simeq c_{x_0}$, then $id_Y = h \circ h^{-1} = h \circ id_X \circ h^{-1} \simeq h \circ c_{x_0} \circ h^{-1}$, which is the constant map of Y sending every point to $h(x_0)$.

3. A (path-connected) space with nontrivial finite fundamental group must be compact.

   No, take any space X with nontrivial finite fundamental group, such as P which has $\pi_1(P, x_0) \cong C_2$, and note that $\pi_1(X \times \mathbb{R}, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(\mathbb{R}, y_0) \cong \pi_1(X, x_0) \times \{1\} \cong \pi_1(X, x_0)$, but $X \times \mathbb{R}$ is not compact.

4. Only finitely many homeomorphism classes of (compact, connected) surfaces can have the same Euler characteristic.

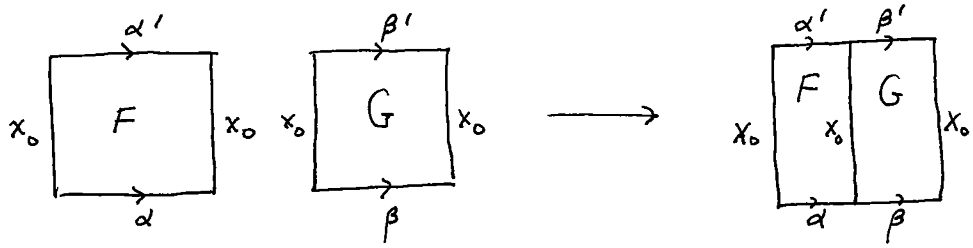
   True, for we have $\chi(S^2\#gT\#nP\#\ell D) = 2 - 2g - n - \ell$, and since $g, n, \ell \geq 0$, there are at most finitely many solutions for a given value of $\chi(F)$.

X. (6) Let $j_0, j_1\colon X \to Y$ be imbeddings. Define what it means to say that j_0 and j_1 are *isotopic.* Define what it means to say that j_0 and j_1 are *ambiently isotopic.*

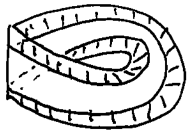
Isotopic means that there is a homotopy $J_t\colon j_0 \simeq j_1$ with each J_t an imbedding. Ambiently isotopic means there is an isotopy of homeomorphisms $H_t\colon id_Y \simeq H_1$ with $H_1 \circ j_0 = j_1$.

XI. (6) Let $\langle\alpha\rangle, \langle\beta\rangle \in \pi_1(X, x_0)$. Show that the multiplication operation on $\pi_1(X, x_0)$ defined by $\langle\alpha\rangle\langle\beta\rangle = \langle\alpha * \beta\rangle$ is well-defined. (You do not need to check continuity of the path homotopy, just describe the path homotopy that verifies well-definedness. A picture might be helpful.)

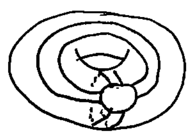
We must show that if $\alpha \simeq_p \alpha'$ and $\beta \simeq_p \beta'$, then $\alpha * \beta \simeq_p \alpha' * \beta'$. If $\alpha_t\colon \alpha \simeq \alpha'$ and $\beta_t\colon \beta \simeq \beta'$ are path homotopies, then $\alpha_t * \beta_t\colon \alpha * \beta \simeq_p \alpha' * \beta'$. A picture of this path homotopy is:



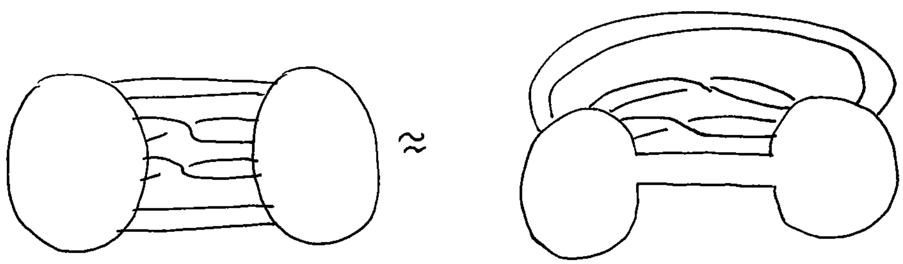
I.



II.



IV.2.



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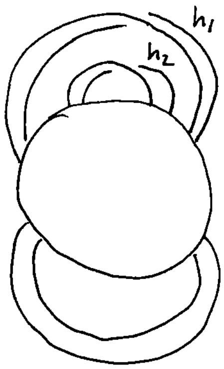

slide right end of h_1 over h_2 twice
(or, slide all of h_2 over h_1 once)

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$= P \# P \# D \# D$