I. Let $M$ be a manifold with boundary. What is a collar of $\partial M$? Draw a picture of a Möbius band, showing a collar on its boundary.

A collar of $\partial M$ is an imbedding $\partial M \times I \to M$ that sends $(x,0)$ to $x$ for all $x \in \partial M$ (or, that sends $(x,1)$ to $x$ for all $x \in \partial M$). For the picture see the last page.

II. Let $X$ be obtained from a disk by attaching two untwisted 1-handles whose ends alternate in the boundary of the disk. Draw a picture of $X$ imbedded in a torus.

See the last page.

III. Prove that every contractible space is path-connected.

Suppose that $X$ is contractible, and let $F: id_X \simeq c_{x_0}$ where $c_{x_0}: X \to X$ is the constant map that sends every point $x$ to $x_0$. For any $x \in X$, define $\alpha: I \to X$ by $\alpha(t) = F(x,t)$. This is path from $x$ to $x_0$ (it is continuous since it is the restriction of $F$ to $\{x\} \times I$ preceded by the inclusion $I \to X \times I$ that sends $t$ to $(x,t)$), so every point of $X$ is in the path component of $x_0$.

IV. A compact connected 2-manifold is shown at the right. It has a handle structure with two 0-handles and four 1-handles.

1. Use Euler characteristic and orientability to determine the homeomorphism type of this 2-manifold.

2. Determine the homeomorphism type directly by using handle slides to simplify this handle structure into a standard form.

We observe that the surface is nonorientable, and has two boundary circles. It Euler characteristic is $2 - 4 = -2$, so we have $2 - n - 2 = -2$ and therefore $n = 2$. So the surface is $P\#P\#D\#D$. For the handle sliding, see the last page.

V. Let $f: S^1 \to S^1$ be the homeomorphism sending $\theta$ to $\theta + \pi$. Construct an explicit isotopy from $id_{S^1}$ to $f$.

Define $F: S^1 \times I \to S^1$ by $F(\theta,t) = \theta + t\pi$. Each $F_t$ is a homeomorphism (with inverse sending $\theta$ to $\theta - t\pi$), $F_0 = id_{S^1}$, and $F_1(\theta) = \theta + \pi$.

VI. Let $\gamma: I \to S^1$ be the path defined by $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$. Verify that $\gamma \ast (\gamma \ast \gamma) \neq (\gamma \ast \gamma) \ast \gamma$.

$\gamma \ast (\gamma \ast \gamma)(1/4) = \gamma(2 \cdot 1/4) = \gamma(1/2) = (-1,0)$, but $(\gamma \ast \gamma) \ast \gamma(1/4) = \gamma \ast \gamma(2 \cdot 1/4) = \gamma(2 \cdot 2 \cdot 1/4) = \gamma(1) = (1,0)$. 
VII. The following is an incorrect proof of the true fact that $\pi_1(S^2, x_0)$ is the trivial group. Find the error in it:

Let $\alpha: I \rightarrow S^2$ be a loop based at $x_0$. Choose a point $x_1 \in S^2$ which is not in the image of $\alpha$. Since $S^2 - \{x_1\}$ is homeomorphic to $\mathbb{R}^2$, and any two loops based at the same point in $\mathbb{R}^2$ are path-homotopic, $\alpha$ is path-homotopic to the constant path at $x_0$. Therefore $\pi_1(S^2, x_0)$ is the trivial group.

If $\alpha$ is surjective, then a point $x_1$ not in the image of $\alpha$ does not exist.

VIII. Use the facts that $\pi_1(S^1, s_0) \cong \mathbb{Z}$ and $\pi_1(D^2, s_0) \cong \{0\}$, together with the functorial properties of the induced homomorphism, to prove that the circle is not a retract of the disk.

Let $i: S^1 \rightarrow D^2$ be the inclusion and suppose for contradiction that there exists a retraction $r: D^2 \rightarrow S^1$, so that $r \circ i = id_{S^1}$. Then $id_{\pi_1(S^1, s_0)} = (id_{S^1})_# = (r \circ i)_# = r_# \circ i_# : \pi_1(S^1, s_0) \rightarrow \pi_1(D^2, s_0) \rightarrow \pi_1(S^1, s_0)$, but this would be a sequence of homomorphisms $\mathbb{Z} \rightarrow \{0\} \rightarrow \mathbb{Z}$ whose composition is the identity.

IX. Prove or give a counterexample:

1. A connected sum $M \# M$ can be homeomorphic to $M$.

   True, for example $S^2\#S^2 = S^2$.

2. The property of being contractible is a topological invariant.

   Yes, if $h: X \rightarrow Y$ is a homeomorphism, and $X$ is contractible with $id_X \simeq c_{x_0}$, then $id_Y = h \circ h^{-1} = h \circ id_X \circ h^{-1} \simeq h \circ c_{x_0} \circ h^{-1}$, which is the constant map of $Y$ sending every point to $h(x_0)$.

3. A (path-connected) space with nontrivial finite fundamental group must be compact.

   No, take any space $X$ with nontrivial finite fundamental group, such as $P$ which has $\pi_1(P, x_0) \cong C_2$, and note that $\pi_1(X \times \mathbb{R}, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(\mathbb{R}, y_0) \cong \pi_1(X, x_0) \times \{1\} \cong \pi_1(X, x_0)$, but $X \times \mathbb{R}$ is not compact.

4. Only finitely many homeomorphism classes of (compact, connected) surfaces can have the same Euler characteristic.

   True, for we have $\chi(S^2\#gT\#nP\#\ell D) = 2 - 2g - n - \ell$, and since $g, n, \ell \geq 0$, there are at most finitely many solutions for a given value of $\chi(F)$.

X. Let $j_0, j_1: X \rightarrow Y$ be imbeddings. Define what it means to say that $j_0$ and $j_1$ are isotopic. Define what it means to say that $j_0$ and $j_1$ are ambiently isotopic.

Isotopic means that there is a homotopy $J_t: j_0 \simeq j_1$ with each $J_t$ an imbedding. Ambiently isotopic means there is an isotopy of homeomorphisms $H_t: id_Y \simeq H_1$ with $H_1 \circ j_0 = j_1$. 
XI. Let \((\alpha), (\beta) \in \pi_1(X, x_0)\). Show that the multiplication operation on \(\pi_1(X, x_0)\) defined by \((\alpha)(\beta) = (\alpha \ast \beta)\) is well-defined. (You do not need to check continuity of the path homotopy, just describe the path homotopy that verifies well-definedness. A picture might be helpful.)

We must show that if \(\alpha \simeq_p \alpha'\) and \(\beta \simeq_p \beta'\), then \(\alpha \ast \beta \simeq_p \alpha' \ast \beta'\). If \(\alpha_t : \alpha \simeq \alpha'\) and \(\beta_t : \beta \simeq \beta'\) are path homotopies, then \(\alpha_t \ast \beta_t : \alpha \ast \beta \simeq_p \alpha' \ast \beta'\). A picture of this path homotopy is:

\[ F \xrightarrow{\alpha'} G \xrightarrow{\beta'} \]

\[ F \times_o G \]

II.

III.

IV. 2.

\[ \approx \]

\[ \approx \]

\[ \approx \]

\[ \approx \]

\[ \approx \]

slide right end of \(h_1\) over \(h_2\) twice (or, slide all of \(h_2\) over \(h_1\), once)

\[ = P \# P \# D \# D \]