Mathematics 5863 Name (please print) ____________________

Examination I

February 22, 2005

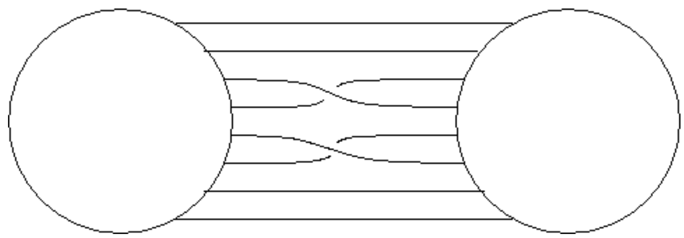
Instructions: Give brief, clear answers.

I. (6) Let M be a manifold with boundary. What is a *collar* of ∂M? Draw a picture of a Möbius band, showing a collar on its boundary.

II. (5) Let X be obtained from a disk by attaching two untwisted 1-handles whose ends alternate in the boundary of the disk. Draw a picture of X imbedded in a torus.

III. (10) Prove that every contractible space is path-connected.

IV. (20) A compact connected 2-manifold is shown at the right. It has a handle structure with two 0-handles and four 1-handles.



1. Use Euler characteristic and orientability to determine the homeomorphism type of this 2-manifold.
2. Determine the homeomorphism type directly by using handle slides to simplify this handle structure into a standard form.

V. (5) Let $f\colon S^1 \to S^1$ be the homeomorphism sending θ to $\theta + \pi$. Construct an explicit isotopy from id_{S^1} to f.

VI. (10) Let $\gamma\colon I \to S^1$ be the path defined by $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$. Verify that $\gamma * (\gamma * \gamma) \neq (\gamma * \gamma) * \gamma$.

VII. (5) The following is an incorrect proof of the true fact that $\pi_1(S^2, x_0)$ is the trivial group. Find the error in it: Let $\alpha\colon I \to S^2$ be a loop based at x_0. Choose a point $x_1 \in S^2$ which is not in the image of α. Since $S^2 - \{x_1\}$ is homeomorphic to $\mathbb{R}^2$, and any two loops based at the same point in $\mathbb{R}^2$ are path-homotopic, α is path-homotopic to the constant path at x_0. Therefore $\pi_1(S^2, x_0)$ is the trivial group.

VIII. (10) Use the facts that $\pi_1(S^1, s_0) \cong \mathbb{Z}$ and $\pi_1(D^2, s_0) \cong \{0\}$, together with the functorial properties of the induced homomorphism, to prove that the circle is not a retract of the disk.

IX. (20) Prove or give a counterexample:

1. A connected sum $M \# M$ can be homeomorphic to M.
2. The property of being contractible is a topological invariant.
3. A (path-connected) space with nontrivial finite fundamental group must be compact.
4. Only finitely many homeomorphism classes of (compact, connected) surfaces can have the same Euler characteristic.

X. (6) Let $j_0, j_1\colon X \to Y$ be imbeddings. Define what it means to say that j_0 and j_1 are *isotopic.* Define what it means to say that j_0 and j_1 are *ambiently isotopic.*

XI. (6) Let $\langle\alpha\rangle, \langle\beta\rangle \in \pi_1(X, x_0)$. Show that the multiplication operation on $\pi_1(X, x_0)$ defined by $\langle\alpha\rangle\langle\beta\rangle = \langle\alpha * \beta\rangle$ is well-defined. (You do not need to check continuity of the path homotopy, just describe the path homotopy that verifies well-definedness. A picture might be helpful.)