Mathematics 5863 Name (please print) ______________________

Final Examination

May 10, 2005

Give brief answers, emphasizing the key point without using too much time on unimportant details.

Even in problems where you cannot do one of the part, try to solve later parts of the problem taking the previous parts as known.

In any problem involving fundamental groups or covering maps, it is assumed that all spaces involved are connected, locally path-connected, semilocally simply-connected Hausdorff spaces.

You may freely use the fact that $\chi(S^2\#gT\#nP\#kD) = 2 - 2g - n - k$.

I. (8) State the Invariance of Domain theorem. Prove that if $j\colon S \to \mathbb{R}^2$ is an imbedding from a 2-manifold S with empty boundary to $\mathbb{R}^2$, then the image of j is open.

The Invariance of Domain theorem says that if $U \subseteq \mathbb{R}^n$ is an open subset, and $f\colon U \to \mathbb{R}^n$ is a continuous injection, then $f(U)$ is open.

For the second part, for each $x \in S$, choose an open neighborhood U_x of x in S with $U_x \approx \mathbb{R}^2$. The restriction $j|_{U_x}\colon U_x \to \mathbb{R}^2$ is a continous injection, so $j(U_x)$ is open in $\mathbb{R}^2$. Therefore $j(S) = j(\cup_{x\in S} U_x) = \cup_{x\in S}\, j(U_x)$ is an open subset of $\mathbb{R}^2$.

II. (8) Let $A \subseteq B \subseteq X$.

1. Prove that if A is a retract of X, then A is a retract of B.

   Let $r\colon X \to A$ be a retraction. Then $r|_B\colon B \to A$ and $r|_B(i(a)) = a$ for each $A \in B$, so $r|_B$ is a retraction.

2. Find an example where A is a deformation retract of X, but A is not a deformation retract of B.

   Let $X = D^2$, $B = \partial D^2 = S^1$, and let $A = \{x_0\}$ for some $x_0 \in B$. Then A is a deformation retraction of X, by a straght-line deformation $F(x,t) = (1-s)x + sx_0$. But A is not a deformation retraction of B, since $\pi_1(A) = \{1\}$ but $\pi_1(B) \cong \mathbb{Z}$.

III. (6) Give an example (with a brief explanation) of two compact, connected surfaces that are homotopy equivalent but not homeomorphic.

Let $F_1 = D\#D\#D$ and $F_2 = T\#D$. Both of these have a one-point union $S^1 \vee S^1$ as a deformation retract (for F_1, it is just a figure-8 that encloses two of the holes, while for F_2 it is the boundary of the square when one regards F_2 as a square with identifications on the boundary and an open disk removed from the interior), so $F_1 \simeq S^1 \vee S^1 \simeq F_2$. But F_1 has three boundary circles and F_2 has only one boundary circle, so they are not homeomorphic.

IV. (10) All surfaces in this problem are assumed to be compact and connected. As usual, $\chi(F)$ means the Euler characteristic of F.

1. Let F be a compact, connected surface. Recall that when one removes the interior of a 2-disk admissibly imbedded in F, the resulting manifold is $F \# D$. Use the Classification Theorem to verify that $\chi(F \# D) = \chi(F) - 1$.

   If $F = S^2 \# gT \# nP \# \ell D$, then $\chi(F) = 2-2g-n-\ell$, while $\chi(F \# D) = 2-2g-n-(\ell+1) = \chi(F)-1$.

2. Let F_1 and F_2 be two surfaces, and let F be a connected surface obtained by identifying a boundary circle of F_1 with a boundary circle of F_2 (for example, when one identifies the boundaries of two disks, one obtains a 2-sphere). Use the Classification Theorem to verify that $\chi(F) = \chi(F_1) + \chi(F_2)$.

   Notice that the number of boundary circles of F is 2 less than the sum of the numbers of boundary circles of F_1 and F_2. So if $F_i = S^2 \# g_iT \# n_iP \# \ell_iD$, we have $F = F_1 \cup F_2 = S^2 \# (g_1 + g_2)T \# (n_1 + n_2)P \# (\ell_1 + \ell_2 - 2)D$, so $\chi(F_1) + \chi(F_2) = (2 - 2g_1 - n_1 - \ell_1) + (2 - 2g_1 - n_1 - \ell_1) = 4 - 2(g_1 + g_2) - (n_1 + n_2) - (\ell_1 + \ell_2) = 2 - 2(g_1 + g_2) - (n_1 + n_2) - (\ell_1 + \ell_2 - 2) = \chi(F)$.

3. Put these two items of information together with the fact that $F_1 \# F_2$ is obtained by identifying boundary circles of $F_1 \# D$ and $F_2 \# D$ to prove that $\chi(F_1 \# F_2) = \chi(F_1) + \chi(F_2) - 2$.

   We have $\chi(F_1 \# F_2) = \chi((F_1 \# D) \cup (F_2 \# D)) = \chi(F_1 \# D) + \chi(F_2 \# D) = \chi(F_1) - 1 + \chi(F_2) - 1 = \chi(F_1) + \chi(F_2) - 2$.

V. (6) Assume the fact that if a map $f\colon X \to Y$ is homotopic to a constant map, then it extends to a map $F\colon C(X) \to Y$, where $C(X)$ is the cone on X, and X is regarded as a subspace of $C(X)$ by identifying x with $(x, 0)$. Prove that S^2 is not contractible.

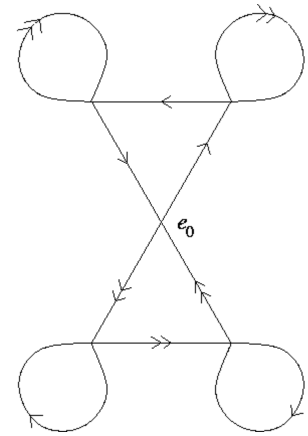
Suppose that $id_{S^2} \simeq c$ for some constant map. By the fact, there exists an extension $F\colon C(S^2) \to S^2$ with $F|_{S^2} = id_{S^2}$. But $C(S^2)$ is homeomorphic to D^3, with S^2 corresponding to ∂D^3 (send $[(x, t)] \in C(S^2) = (S^2 \times I)/(x, 1) \sim (x', 1)$ for all $x, x' \in S^2$ to $(1 - t)x \in D^3 \subset \mathbb{R}^3$), so this violates the No Retraction Theorem.

VI. (6) Let X and Y be spaces, and give the set of continuous functions $\mathcal{C}(X, Y)$ the compact-open topology. Show that if Y is Hausdorff, then $\mathcal{C}(X, Y)$ is Hausdorff.

Let $f, g \in \mathcal{C}(X, Y)$ with $f \neq g$. Then for some $x_0 \in X$, $f(x) \neq g(x)$. Let U and V be disjoint neighborhoods of $f(x_0)$ and $g(x_0)$ in Y. Then $f \in S(\{x_0\}, U)$, $g \in S(\{x_0\}, V)$, and $S(\{x_0\}, U) \cap S(\{x_0\}, V)$ is empty since no function can take x_0 into two disjoint subsets of Y.

VII. (6) Give the set of continuous functions $\mathcal{C}(\mathbb{R}, \mathbb{R})$ from $\mathbb{R}$ to $\mathbb{R}$ the compact-open topology. For $n \in \mathbb{N}$, define $f_n \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ by $f_n(x) = 0$ for $x \leq n$, $f_n(x) = x - n$ for $n \leq x \leq n + 1$, and $f_n(x) = 1$ for $n + 1 \leq x$. Prove that the sequence $\{f_n\}$ converges to the zero function.

Let $\cap_{i=1}^{n} S(C_i, U_i)$ be a basic open set that contains the zero function g. Then $g(C_i) = \{0\} \subset U_i$ for each i. Since $\cup_{i=1}^{n} C_i$ is compact, it is contained in $[-N, N]$ for some positive integer N. Then if $n \geq N$, $f_n(C_i) \subseteq f_N([-N, N]) = \{0\} \subset U_i$ for each i, so $f_n \in \cap_{i=1}^{n} S(C_i, U_i)$.



VIII. (8) A covering space E of $B = S^1 \vee S^1$ is shown at the right. As usual, the single arrows cover the circle which corresponds to an element $a \in \pi_1(B, b_0)$, and the double arrows cover the other circle, which corresponds to an element $b \in \pi_1(B, b_0)$. Write G for $\pi_1(B, b_0)$ and H for $p_\#(\pi_1(E, e_0))$.

1. Determine the number n of right cosets of H in G, and find elements $g_1, \ldots, g_n$ in G such that the cosets are $Hg_1, \ldots, Hg_n$.

   $p^{-1}(b_0)$ consists of five points, so there are five cosets. The lifts of the five elements 1, a, a^2, b, and b^2 end at the five different points of $p^{-1}(b_0)$, so the five cosets of H can be written as H, Ha, Ha^2, Hb, and Hb^2.

2. What is the group of covering transformations for this covering space? Why?

   It is the trivial group. No covering transformation can take e_0 to one of the other four points of $p^{-1}(b_0)$, since each of them is contained in a closed loop projecting to b or to a, but e_0 is not contained in any such loop, so the only covering transformation is id_E.

IX. (6) Let X, Y, and Z be locally compact Hausdorff spaces, and let $C\colon \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ be the composition function defined by $C(f,g) = g \circ f$. Show that if $g \circ f \in S(C,U) \subseteq \mathcal{C}(X,Z)$, then there exists a basic open set W in $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z)$ such that $(f,g) \in W$ and $C(W) \subseteq S(C,U)$.

$g^{-1}(U)$ is an open subset of Y, and since $g \circ f(C) \subseteq U$, we must have the compact subset $f(C) \subseteq g^{-1}(U)$. Since Y is locally compact Hausdorff, there exists an open set V in Y, with $\overline{V}$ compact, such that $f(C) \subseteq V \subseteq \overline{V} \subseteq g^{-1}(U)$. We have $(f,g) \in S(C,V) \times S(\overline{V},U)$. Moreover, if $(f',g') \in S(C,V) \times S(\overline{V},U)$, then $C(f',g')(C) = g'(f'(C)) \subseteq g'(\overline{V}) \subseteq U$. Therefore $C(S(C,V) \times S(\overline{V},U)) \subseteq S(C,U)$.

X. (8) Let $X = \cup_{i=1}^{\infty} U_i$, where each U_i is open and each $U_i \subseteq U_{i+1}$.

1. Prove that if $\alpha\colon (I, \partial I) \to (X, x_0)$ represents an element of $\pi_1(X, x_0)$, then $\alpha(I) \subseteq U_N$ for some N.

   $\alpha(I)$ is a compact subset of X, contained in the union $\cup_{i=1} U_i$, so $\alpha(I) \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$ for some finite subcollection of the U_i. Letting N be the largest of the i_k, we have $\alpha(I) \subseteq U_N$.

2. Prove that if each U_i is simply-connected, then X is simply-connected.

   First we show that X is path-connected. Let $x_0, x_1 \in X$. Then $x_0, x_1 \in U_N$ for some N (each is in some U_n, take the larger n). Since U_N is simply connected, it is path-connected. A path in U_N from x_0 to x_1 is also a path in X from x_0 to x_1.

   Now, to show that $\pi_1(X, x_0)$ is trivial, let $\langle \alpha \rangle \in \pi_1(X, x_0)$. Then $\alpha(I) \subseteq U_N$ for some N. Therefore $\alpha = i \circ \alpha'$, where $\alpha'\colon I \to U_N$ is obtained from α by restriction of the range. But then, $\langle \alpha \rangle = \langle i \circ \alpha' \rangle = i_\#(\langle \alpha' \rangle)$, where $\langle \alpha' \rangle \in \pi_1(U_N, x_0)$. Since U_N is simply-connected, $\pi_1(U_N, x_0) = \{1\}$ and therefore $\langle \alpha \rangle = i_\#(1) = 1$.

XI. (5) State the Lifting Criterion for covering maps.

Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map, and let $f\colon (X, x_0) \to (B, b_0)$ be continuous. Then there exists a lift $F\colon (X, x_0) \to (E, e_0)$ of f (that is, a map such that $p \circ F = f$) if and only if $f_\# \pi(X, x_0) \subseteq p_\# \pi_1(E, e_0)$.

XII. (10) Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map. Prove that a loop α in B based at b_0 lifts to a loop in E at e_0 if and only if $\langle \alpha \rangle \in p_{\#}\, (\pi_1(E, e_0))$.

Let $\widetilde{\alpha}$ be the lift of α starting at e_0. If $\widetilde{\alpha}$ is a loop, then $p_{\#}(\langle \widetilde{\alpha} \rangle) = \langle p \circ \widetilde{\alpha} \rangle = \langle \alpha \rangle$, so $\langle \alpha \rangle \in p_{\#}\, (\pi_1(E, e_0))$. Conversely, suppose that $\langle \alpha \rangle = p_{\#}(\langle \beta \rangle) = \langle p \circ \beta \rangle$ for some $\langle \beta \rangle \in \pi_1(E, e_0$. Then $\alpha \simeq_p p \circ \beta$, so (since β is the unique lift of $p \circ \beta$ starting at e_0) $\widetilde{\alpha} \simeq_p \beta$, and therefore $\widetilde{\alpha}(1) = \beta(1) = e_0$.

XIII. (6) Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be covering maps with $\widetilde{X}$ and $\widetilde{Y}$ simply-connected. Prove that if $f\colon X \to Y$ is any map, then there exists a map $F\colon \widetilde{X} \to \widetilde{Y}$ such that $q \circ F = f \circ p$.

Choose a basepoint $x_0 \in \widetilde{X}$, and any point $y_0 \in \widetilde{Y}$ with $q(y_0) = f \circ q(x_0)$. We have $(f \circ p)_{\#}(\pi_1(\widetilde{X}, x_0)) = (f \circ p)_{\#}(\{1\}) \subseteq q_{\#}(\pi_1(\widetilde{Y}, y_0))$, so there exists a lift $F\colon (\widetilde{X}, x_0) \to (\widetilde{Y}, y_0)$ of $f \circ p$. That is, $q \circ F = f \circ p$.