26. (3/8) Let \( \alpha : I \to S^1 \) be a path. Let \( \tilde{\beta}_1 \) and \( \tilde{\beta}_2 \) be two lifts of \( \alpha \) to \( \mathbb{R} \). Prove that for some \( N \in \mathbb{Z} \), \( \tilde{\beta}_2(t) = \tilde{\beta}_1(t) + N \) for all \( t \in I \) (let \( N = \tilde{\beta}_2(0) - \tilde{\beta}_1(0) \) and define \( \tau(r) = r + N \), check that \( p \circ \tau = p \), and use uniqueness of path lifting). Deduce that \( \tilde{\beta}_1(1) - \tilde{\beta}_1(0) = \tilde{\beta}_2(1) - \tilde{\beta}_2(0) \).

Let \( N = \tilde{\beta}_2(0) - \tilde{\beta}_1(0) \). Since \( p(\tilde{\beta}_1(0) - \tilde{\beta}_1(0)) = \exp(2\pi i \tilde{\beta}_2(0) - 2\pi i \tilde{\beta}_1(0)) = \exp(2\pi i \tilde{\beta}_2(0))/\exp(2\pi i \tilde{\beta}_1(0)) = p(\tilde{\beta}_2(0))/p(\tilde{\beta}_1(0)) = \alpha(0)/\alpha(0) = 1 \), \( N \) is an integer. Define \( \tau : \mathbb{R} \to \mathbb{R} \) by \( \tau(r) = r + N \), so that \( p \circ \tau(r) = p(r + N) = p(r) \), that is, \( p \circ \tau = p \). We have \( p \circ \tau \circ \tilde{\beta}_1 = p \circ \tilde{\beta}_1 = \alpha \), and \( \tau \circ \tilde{\beta}_1(0) = \tilde{\beta}_2(0) \), so by uniqueness of lifts, \( \tau \circ \tilde{\beta}_1 = \tilde{\beta}_2 \), that is, \( \tilde{\beta}_2(t) = \tilde{\beta}_1(t) + N \) for all \( t \). In particular, for \( t = 1 \) we have \( \tilde{\beta}_2(1) = \tilde{\beta}_1(1) + \tilde{\beta}_2(0) - \tilde{\beta}_1(0) \), so \( \tilde{\beta}_1(1) - \tilde{\beta}_1(0) = \tilde{\beta}_2(1) - \tilde{\beta}_2(0) \).

27. (3/8) Prove that \( q : Z \times \mathbb{R} \to S^1 \) defined by \( q(n, r) = p(r) \) has unique path lifting and unique homotopy lifting. (Let \( \alpha : I \to S^1 \) and let \( (n, r_0) \in Z \times \mathbb{R} \) with \( q(n, r_0) = \alpha(0) \). By unique path lifting for \( \mathbb{R} \to S^1 \), there exists \( \tilde{\alpha}_1 : I \to \mathbb{R} \) with \( p \circ \tilde{\alpha}_1(t) = \alpha(t) \). Use \( \tilde{\alpha}_1 \) to define the lift \( \tilde{\alpha} \). To prove that the \( \tilde{\alpha} \) is unique, let \( p_1 : Z \times \mathbb{R} \to Z \) and \( p_2 : Z \times \mathbb{R} \to \mathbb{R} \) be the projection maps, and show that \( p_1 \circ \tilde{\alpha} \) and \( p_2 \circ \tilde{\alpha} \) are uniquely determined.)

Let \( \alpha : I \to S^1 \) and let \( (n, r_0) \in Z \times \mathbb{R} \) with \( q(n, r_0) = \alpha(0) \). By unique path lifting for \( \mathbb{R} \to S^1 \), there exists \( \tilde{\alpha}_1 : I \to \mathbb{R} \) with \( p \circ \tilde{\alpha}_1(t) = \alpha(t) \). Define \( \tilde{\alpha} : I \to Z \times \mathbb{R} \) by \( \tilde{\alpha}(t) = (n, \tilde{\alpha}_1(t)) \); this is a lift of \( \alpha \). To prove that it is unique, suppose that \( \tilde{\gamma} : I \to Z \times \mathbb{R} \) is any lift of \( \alpha \) starting at \( (n, r_0) \). Let \( p_1 : Z \times \mathbb{R} \to Z \) and \( p_2 : Z \times \mathbb{R} \to \mathbb{R} \) be the projection maps. Now \( p_1 \circ \tilde{\gamma} : I \to Z \), and since the maximal connected subsets of \( Z \) are points, \( p_1 \circ \tilde{\gamma} \) must be the constant map to \( n \). On the other hand, \( p_2 \circ \tilde{\gamma} : I \to \mathbb{R} \), and \( p_2 \circ \tilde{\gamma} = q \circ \tilde{\gamma} = \alpha \), so by uniqueness of lifts to \( \mathbb{R} \), \( p_2 \circ \tilde{\gamma} = \tilde{\alpha}_1 \). Since \( p_1 \circ \tilde{\alpha} = p_1 \circ \tilde{\alpha} \) and \( p_2 \circ \tilde{\alpha} = p_2 \circ \tilde{\gamma} = p_2 \circ \tilde{\alpha} \), we have \( \tilde{\gamma} = \tilde{\alpha} \).

The proof for unique lifting of homotopies is very similar.

28. (3/8) Prove that \( q_n : S^1 \to S^1 \) defined by \( q_n(z) = z^n \) (where \( z \in \mathbb{C} \)) has unique path lifting and unique homotopy lifting. Hint: do not repeat the proof of these results for \( p : \mathbb{R} \to S^1 \) Define \( p_n : \mathbb{R} \to S^1 \) by \( p_n(r) = p(r/n) \) and use the facts that \( p = q_n \circ p_n \) and that \( p \) has unique path lifting and unique homotopy lifting.

Let \( \alpha : I \to S^1 \) and suppose that \( s_0 \in S^1 \) with \( p(s_0) = \alpha(0) \). Define \( p_n : \mathbb{R} \to S^1 \) by \( p_n(r) = p(r/n) \), so that \( q_n \circ p_n(r) = (e^{2\pi i r/n})^n = e^{2\pi i r} = p(r) \), that is, \( q_n \circ p_n = p \). Choose \( r_0 \in \mathbb{R} \) with \( p_n(r_0) \). By unique lifting for \( p \), there exists \( \tilde{\alpha} : I \to \mathbb{R} \) so that \( p \circ \tilde{\alpha} = \alpha \). Then, we have \( p_n \circ \tilde{\alpha} : I \to S^1 \) with \( q_n \circ p_n \circ \tilde{\alpha} = p \circ \tilde{\alpha} = \alpha \), and \( p_n \circ \tilde{\alpha}(0) = p_n(r_0) = s_0 \), proving existence of lifts for \( q_n \).

For uniqueness, suppose that \( \tilde{\alpha}_1, \tilde{\alpha}_2 : I \to S^1 \) are two lifts of \( \alpha \) taking 0 to \( s_0 \). Define \( s_n : \mathbb{R} \to \mathbb{R} \) by \( s_n(r) = nr \), so that \( p_n \circ s_n = p \) and \( p_n \circ s_n(r_0/n) = s_0 \). By uniqueness of lifts for \( p \), for each of \( i = 1, 2 \) there exists a unique \( \tilde{\alpha}_i : I \to \mathbb{R} \) for
30. Let $A$ be a subspace of $X$, and $i: A \to X$ the inclusion map. Recall that a retraction $r: X \to A$ is a map such that $r \circ i = id_A$. Define $r$ to be a deformation retraction if there is a homotopy $F: id_X \simeq i \circ r$ with $F(a, t) = a$ for all $t$ and all $a \in A$. (Note: this is sometimes called a strong deformation retraction.) If there exists a deformation retraction from $X$ to $A$, we say that $A$ is a deformation retract of $X$.

1. Show that each $X \times \{t_0\}$ is a deformation retract of $X \times I$ (most of it is just showing that each $t_0$ is a deformation retract of $I$).

A deformation retract of $I$ to $\{t_0\} \subset I$ is defined by $R(t, s) = (1 - s)t + st_0$.

Now, define $F: X \times I \to X$ by $F((x, t), s) = (x, R(t, s))$.

2. Show that the center circle of a Möbius band is a deformation retract of the Möbius band.

Regard the Möbius band $M$ as the square $I \times I$ with identifications $(0, y) \sim (1, 1 - y)$. Note that the center circle is the subset $(I \times \{1/2\})/(0, 1/2) \sim (1, 1/2)$. Define a deformation retraction $F: I \times I \to I \times I$ by $F((x, y), t) = (x, (1 - t)y + t/2)$. This is a deformation retraction of $I \times I$ to $I \times \{1/2\}$.

To check that it produces a well-defined map on $M \times I$, we observe that $F((0, y), t) = (0, (1 - t)y + t/2) \sim (1, 1 - (1 - t)y - t/2) = (1, (1 - t)(1 - y) + t/2) = F((1, 1 - y), t)$ for all $t, y$. So $F$ preserves identified points and therefore it induces a deformation retraction $F: M \times I \to M$ onto the center circle.
3. Show that if $A$ is a deformation retract of $X$, then $i_\# : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism for each basepoint $a_0 \in A$.

We will show that $i_\# : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism. We have $id_A = r \circ i$, so $id_{\pi_1(A, a_0)} = r_\# \circ i_\#$, showing that $i_\#$ is injective. To see that $i_\#$ is surjective, let $\langle \alpha \rangle \in \pi_1(X, a_0)$. Define $G : I \times I \to X$ by $G(t, s) = F(\alpha(t), s)$. Then $G(t, 0) = F(\alpha(t), 0) = \alpha(t)$, $G(0, s) = F(\alpha(0), s) = F(a_0, s) = a_0$ and similarly $G(1, s) = a_0$, and $G(t, 1) = F(\alpha(t), 1) = i \circ r(\alpha(t)) \in A$. Since $r \circ \alpha(t) \in A$, $i^{-1} \circ r \circ \alpha(t)$ is defined. Letting $\beta = i^{-1} \circ r \circ \alpha$, we have $i_\#(\langle \beta \rangle) = \langle i \circ \beta \rangle = \langle i \circ i^{-1} \circ r \circ \alpha \rangle = \langle r \circ \alpha \rangle = \langle \alpha \rangle$. 