Math 5863 homework

33. (3/22) For \( n \geq 2 \), the dihedral group of order \( 2n \) is the group \( D_n \) consisting of all pairs \( \alpha^i\beta^j \) where \( i \) is an integer modulo \( n \) and \( j \) is an integer modulo 2, with the multiplication rule that \( \alpha^i\beta^j\alpha^k\beta^\ell = \alpha^{i+(-1)^j k}\beta^{j+\ell} \) (that is, \( \beta\alpha^i\beta^{-1} = \alpha^{-i} \)). Verify the following:

1. Check that the condition \( \alpha^i\beta^j\alpha^k\beta^\ell = \alpha^{i+(-1)^j k}\beta^{j+\ell} \) implies that \( \beta\alpha\beta^{-1} = \alpha^{-1} \), and that the condition that \( \beta\alpha\beta^{-1} = \alpha^{-1} \) implies that \( \alpha^i\beta^j\alpha^k\beta^\ell = \alpha^{i+(-1)^j k}\beta^{j+\ell} \).

2. \( D_n \) has \( 2n \) elements.
3. \( D_1 \) is isomorphic to \( C_2 \).
4. \( D_2 \) is isomorphic to \( C_2 \times C_2 \).
5. \( D_n \) is nonabelian for \( n \geq 3 \).
6. The powers of \( \alpha \) form a subgroup isomorphic to \( C_n \).
7. The powers of \( \beta \) form a subgroup isomorphic to \( C_2 \).
8. Find the conjugacy class of each element of \( D_n \).

34. (3/22) Recall that the group \( \text{Isom}_+ (\mathbb{R}^2) \) of orientation-preserving isometries consists of all compositions \( T_v R_\alpha \), for \( v \in \mathbb{R}^2 \) and \( \alpha \in S^1 \) (where we regard \( S^1 \) as the additive group of real numbers modulo \( 2\pi \)), with multiplication given by \( T_v R_\alpha T_w R_\beta = T_{v+R_\alpha(w)} R_{\alpha+\beta} \). Note that the inverse of \( T_v R_\alpha \) is \( R_{-\alpha} T_{-v} \), which is also equal to \( T_{R_{-\alpha}(-v)} R_{-\alpha} \).

1. Verify that the conjugacy class of \( T_v \) (\( v \neq 0 \)) is \( \{ T_w \mid \| w \| = \| v \| \} \). Describe these elements geometrically.
2. Verify that the conjugacy class of \( R_\alpha \) (\( \alpha \neq 0 \)) is \( \{ T_v R_\alpha \mid v \in \mathbb{R}^2 \} \). Show that these elements are exactly the isometries that rotate the plane through an angle \( \alpha \) about some fixed point. (Observe that each conjugate can be written in the form \( T_w R_\alpha T_{-w} \), and think about its geometric effect on the plane.)