Math 5863 homework solutions

19. Prove that two paths $\alpha, \beta\colon I \to \mathbb{R}^n$ are path-homotopic if and only if they have the same starting point and the same ending point.

    If $F\colon \alpha \simeq_p \beta$, then $\alpha(0) = F(t,0) = F(t,1) = \beta(0)$, and similarly $\alpha(1) = \beta(1)$. Conversely, suppose that α and β have the same starting point and the same ending point. Define $F\colon I \times I \to \mathbb{R}^n$ by $F(t,s) = (1-s)\alpha(t) + s\beta(t)$. One checks easily that $F\colon \alpha \simeq_p \beta$. We remark that this applies just as well to paths in any convex subset of $\mathbb{R}^n$.

20. Let X be a path-connected space. We say that X is *simply-connected* if every two paths in X that have the same starting point and same ending point are path-homotopic. Prove that X is simply-connected if and only if every map from S^1 to X extends to a map from D^2 to X. Hint: Use the quotient map from $I \times I$ to D^2 that maps $\{0\} \times I$ to -1 and $\{1\} \times I$ to 1. For sufficiency, given a map from S^1 to X let α be its restriction to the upper half-circle and let β be its restriction to the bottom half-circle.

    Suppose first that X is simply-connected. Let $f\colon S^1 \to X$ be a map of the circle. Define paths $\alpha, \beta\colon I \to X$ by $\alpha(t) = f(\cos(\pi(1-t)), \sin(\pi(1-t)))$ and $\beta(t) = f(\cos(\pi(1-t)), \sin(-\pi(1-t)))$ for $0 \leq t \leq 1$. We see that $\alpha(0) = \beta(0) = f(-1,0)$ and $\alpha(1) = \beta(1) = f(1,0)$. Since X is simply-connected, there is a path homotopy $F\colon \alpha \simeq_p \beta$. There is a quotient map $q\colon I \times I \to D^2$ that collapses $\{0\} \times I$ to $(-1,0)$ and $\{1\} \times I$ to $(1,0)$; more precisely, we can define q by $q(t,s) = (\cos(\pi(1-t)), \sin(\pi(1-t))\sin(\pi(1/2-s)))$. Since F is constant on $q^{-1}(-1,0) = \{0\} \times I$ and $q^{-1}(1,0) = \{1\} \times I$, there is an induced map $\overline{F}\colon D^2 \to X$. To check that $\overline{F}$ extends f, we have $\overline{F}(\cos(\pi(1-t)), \sin(\pi(1-t))) = \overline{F} \circ q(t,0) = F(t,0) = \alpha(t) = f(\cos(\pi(1-t)), \sin(\pi(1-t)))$ and $\overline{F}(\cos(\pi(1-t)), -\sin(\pi(1-t))) = \overline{F} \circ q(t,1) = F(t,1) = \beta(t) = f(\cos(\pi(1-t)), -\sin(\pi(1-t)))$.

    Conversely, suppose that every map from S^1 to X extends to D^2. Suppose that α and β are paths from x_0 to x_1 in X. Define $f\colon S^1 \to X$ by $f(\cos(\pi(1-t)), \sin(\pi(1-t))) = \alpha(t)$ and $f(\cos(\pi(1-t)), \sin(-\pi(1-t))) = \beta(t)$ for $0 \leq t \leq 1$; this is well-defined since $f(\cos(\pi(1-0)), \sin(\pi(1-0))) = \alpha(0) = \beta(0) = f(\cos(\pi(1-0)), \sin(\pi(1-0)))$ and $f(\cos(\pi(1-1)), \sin(\pi(1-1))) = \alpha(1) = \beta(1) = f(\cos(\pi(1-0)), \sin(-\pi(1-0)))$ (and is continuous by gluing on locally finite closed covers). Let $F\colon D^2 \to X$ be an extension of f. For the quotient map $q\colon I \times I \to D^2$ given above, it is straightforward to check that $q \circ F\colon \beta \simeq_p \alpha$.

21. Prove that a path-connected space X is simply-connected if and only if $\pi_1(X, x_0) = \{1\}$ for every choice of basepoint in X.

> Assume first that X is simply-connected. Let $\langle \alpha \rangle \in \pi_1(X, x_0)$. Then α and c_{x_0} are both loops at x_0, so since X is simply-connected, they are path-homotopic. Therefore $\langle \alpha \rangle = \langle c_{x_0} \rangle$, so $\pi_1(X, x_0)$ has only one element.
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> Conversely, assume that $\pi_1(X, x_0) = \{1\}$ for each x_0. Suppose that α and β are paths from x_0 to x_1 in X. Then $\alpha * \overline{\beta}$ is a loop at x_0, and sine $\pi_1(X, x_0)$ has only one element, $\alpha * \overline{\beta} \simeq_p c_{x_0}$. Using properties of path homotopy developed in class, we have $\alpha \simeq_p \alpha * c_{x_1} \simeq_p \alpha * \overline{\beta} * \beta \simeq_p c_{x_0} * \beta \simeq_p \beta$.

22. Prove that any contractible space is simply-connected.

> Let X be contractible, so that there is a homotopy $H\colon id_X \simeq c_{x_0}$ for some constant map c_{x_0}. To show that X is path-connected, let $x \in X$. Then $\alpha(t) = H(x, t)$ is a path in X from x to x_0, so X has only one path component. To show that X is simply-connected, let $f\colon S^1 \to X$ be given. We have $f = id_X \circ f \simeq c_{x_0} \circ f$, which is constant. By problem #7, f extends to $C(S^1)$, which is homeomorphic to D^2. By problem #20, this shows that X is simply-connected.