Math 5863 homework

19. (2/22) Prove that two paths $\alpha, \beta\colon I \to \mathbb{R}^n$ are path-homotopic if and only if they have the same starting point and the same ending point.

20. (2/22) Let X be a path-connected space. We say that X is *simply-connected* if every two paths in X that have the same starting point and same ending point are path-homotopic. Prove that X is simply-connected if and only if every map from S^1 to X extends to a map from D^2 to X. Hint: Use the quotient map from $I \times I$ to D^2 that maps $\{0\} \times I$ to -1 and $\{1\} \times I$ to 1. For sufficiency, given a map from S^1 to X let α be its restriction to the upper half-circle and let β be its restriction to the bottom half-circle.

21. (2/22) Prove that a path-connected space X is simply-connected if and only if $\pi_1(X, x_0) = \{1\}$ for every choice of basepoint in X.

22. (2/22) Prove that any contractible space is simply-connected.

23. (3/1) Let G be a group. For an element g group G, define *conjugation by* g to be the function $\mu(g)\colon G \to G$ that sends x to gxg^{-1}.
    1. Check that $\mu(1) = id_G$ and $\mu(g_1 g_2) = \mu(g_1)\mu(g_2)$. Deduce that $\mu(g)$ is an isomorphism of G.
    2. Define $\mathrm{Aut}(G)$ to be the set of automorphisms of G. Check that $\mathrm{Aut}(G)$ is a group under the operation of composition.
    3. Define $\mathrm{Inn}(G)$ to be the set of inner automorphisms of G. Check that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

24. (3/1) Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ denote the projections. Show that $(p_1)_\# \times (p_2)_\#\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ is an isomorphism.

25. (3/1) Let x_0 and x_1 be two points in the same path-component of X. For a path $\gamma\colon I \to X$ from x_0 to x_1, define $h_\gamma\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $h_\gamma(\langle\alpha\rangle) = \langle\gamma * \alpha * \overline{\gamma}\rangle$.
    1. Show that h_γ is a well-defined homomorphism.
    2. Show that if $\gamma(1) = \tau(0)$, then $h_{\gamma * \tau} = h_\gamma h_\tau$.
    3. Show that if γ is a loop at x_1, then $h_\gamma = \mu(\langle\gamma\rangle)$.
    4. Show that if $\gamma_1 \simeq_p \gamma_2$, then $h_{\gamma_1} = h_{\gamma_2}$. Deduce that h_γ is an isomorphism with inverse $h_{\overline{\gamma}}$. Thus, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic as long as x_0 and x_1 are in the same path component of X.
    5. Show that if β is another path from x_0 to x_1, then $h_\beta^{-1} \circ h_\alpha = \mu(\langle\overline{\beta} * \alpha\rangle)$.
    6. Deduce that if $\pi_1(X, x_1)$ is abelian, then $h_\alpha\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ is independent of the choice of path α from x_0 to x_1.

As a consequence of the previous problem, all choices of α give the same isomorphism h_α when X is path-connected and $\pi_1(X, x_1)$ is abelian. That is, there is a way to identify $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ that is independent of all choices, so in this situation, one may safely write $\pi_1(X)$.