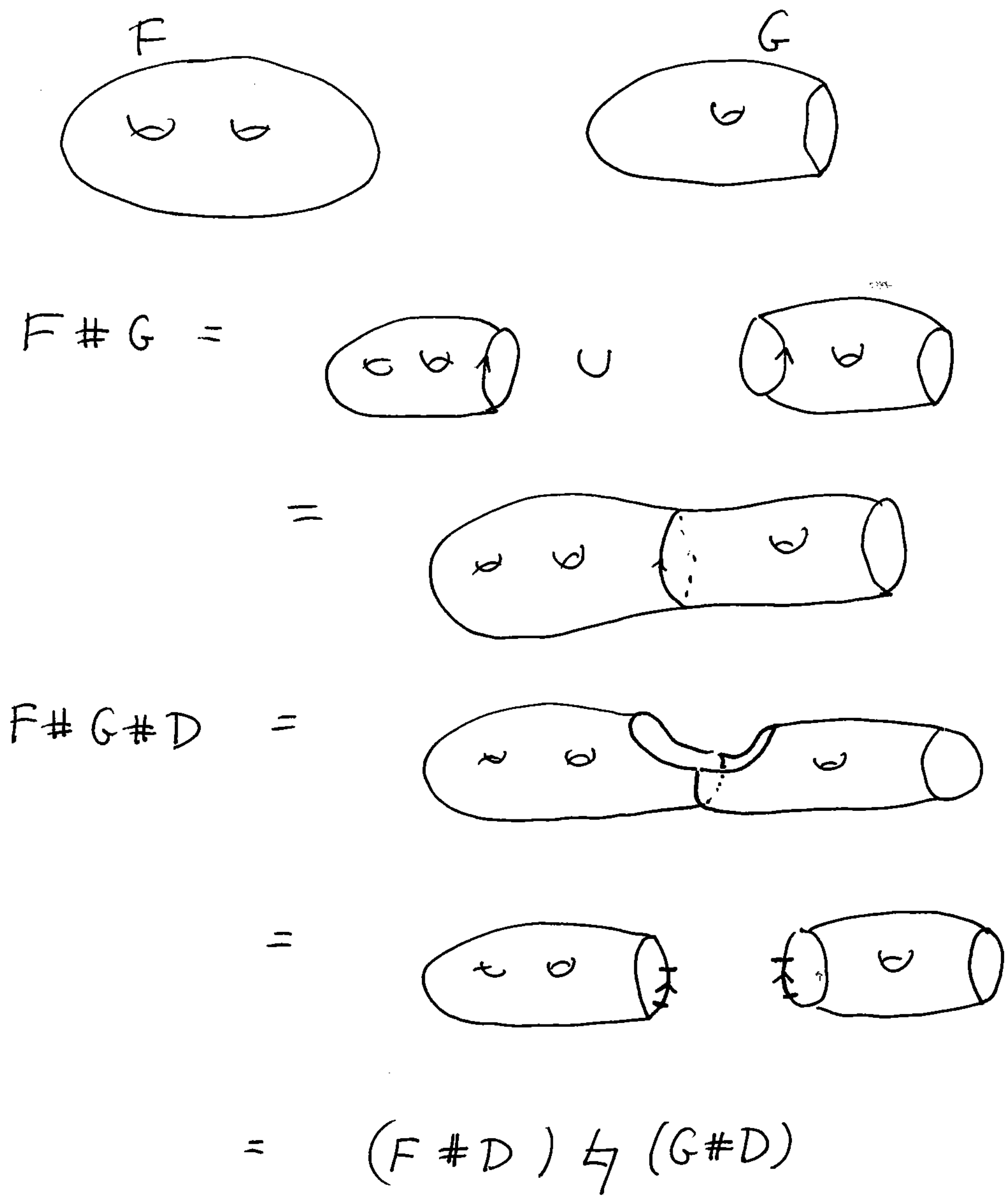
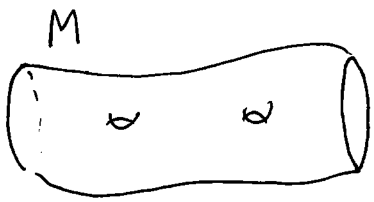
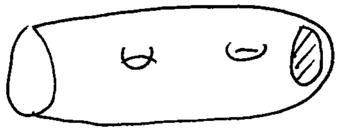
Math 5863 homework solutions

12. (2/8) Let F and G be compact connected 2-manifolds with nonempty boundary. Let α be an arc (an imbedded copy of I) in ∂F, and β an arc in ∂G. Define the *boundary-connected sum* of F and G to be the surface $F \natural G$ obtained from $F \cup G$ by identifying α and β by a homeomorphism. Draw pictures to explain why $(F\#D^2)\natural(G\#D^2) = F\#G\#D^2$.



13. (2/8) Use the previous problem to give a quick explanation of why attaching a twisted 1-handle to a boundary circle of M produces $M\#P$, and why attaching two untwisted 1-handles with alternating ends to a boundary circle of M produces $M\#T$.

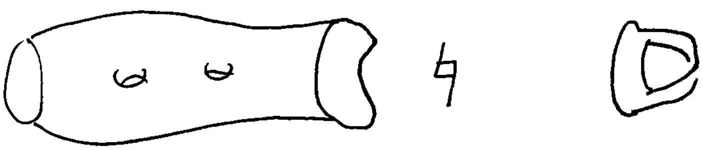


Let $\hat{M} = M \cup D =$  ,

So that $M = \hat{M} \# D$.

$M \cup$ twisted 1-handle

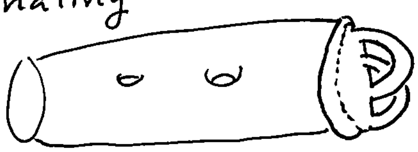
$=$ 

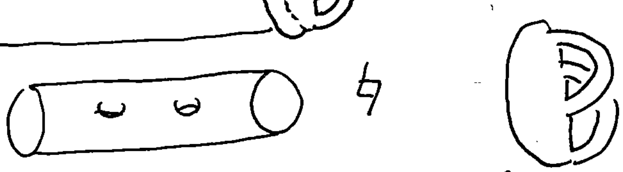
$=$ 

$= (\hat{M} \# D) \sqcup (P \# D)$

$= \hat{M} \# P \# D = (\hat{M} \# D) \# P = M \# P$

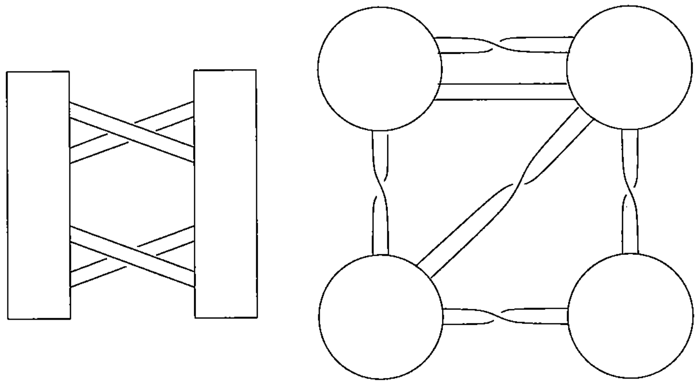
$M \cup$ alternating ~~1-ha~~ untwisted 1-handles

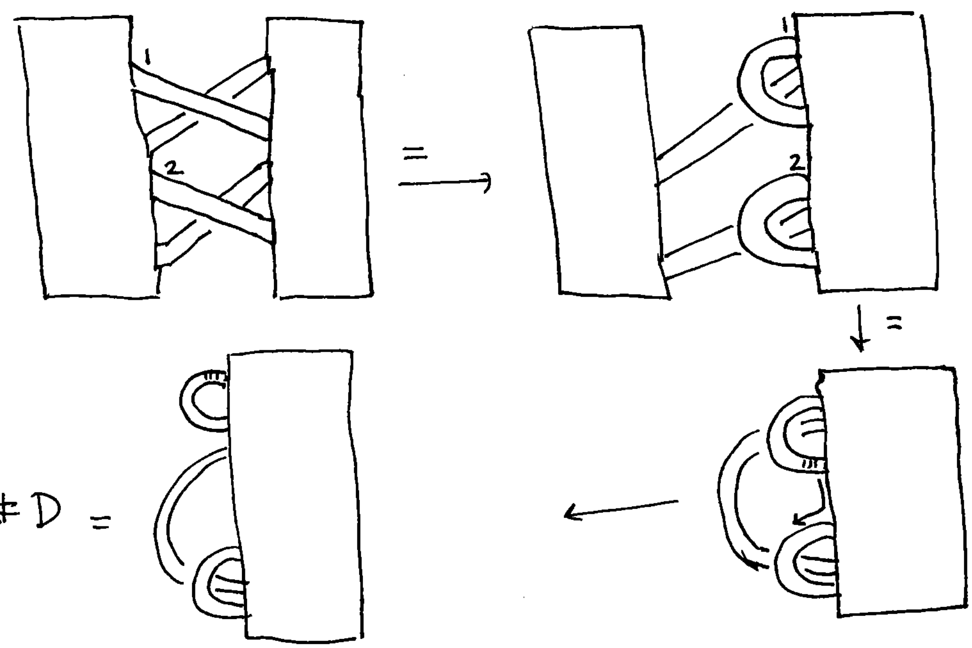
$=$ 

$=$ 

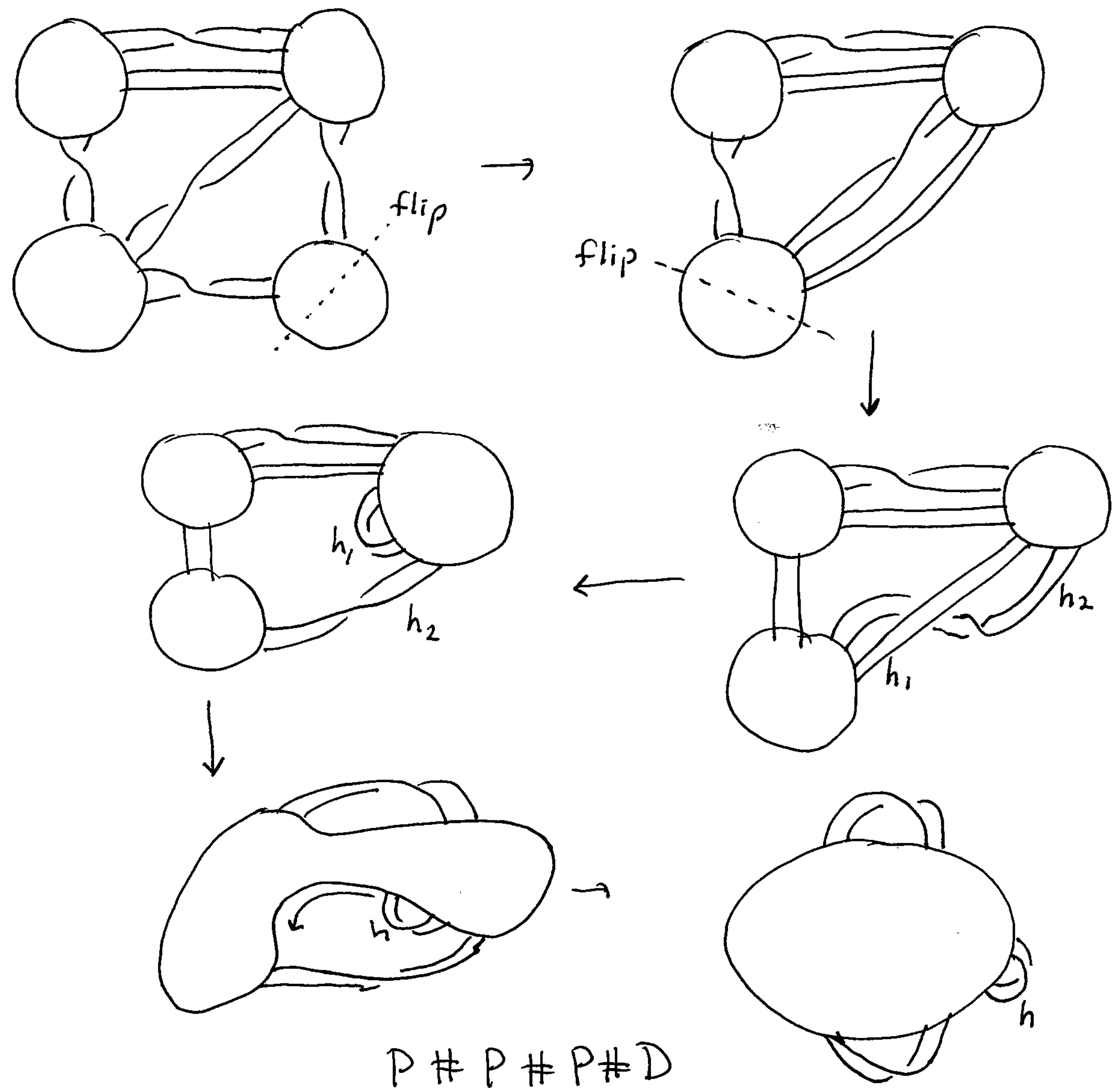
$= (\hat{M} \# D) \sqcup (T \# D) = \hat{M} \# D \# T = M \# T$

14. (2/8) Use handle slides to simplify and identify each of these two surfaces:





$T \# D \# D =$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P # P # P # D

15. (2/8) A space X is defined to be *contractible* if the identity map of X is homotopic to a constant map. Prove the following:

1. If C is a convex subset of $\mathbb{R}^n$ (in particular, C could be $\mathbb{R}^n$ or D^n), then C is contractible.

    Let $f, g\colon X \to C$ be any two maps. The function $F\colon X \times I \to C$ defined by $F(x,t) = (1-t)\,f(x) + t\,g(x)$ defines a homotopy from f to g. That is, any two maps from a space X into a convex subset of $\mathbb{R}^n$ are homotopic. In particular, id_C and c are homotopic for any constant map $c\colon C \to C$, so C is contractible.

2. X is contractible if and only if there is a retraction from $C(X)$ to X.

    By problem 7, id_X is homotopic to a constant map if and only if there exists a map $C(X) \to X$ whose restriction to X is id_X.

3. If X is contractible, then any map from X to any space Y is homotopic to a constant map.

    Since X is contractible, $id_X \simeq c$ for some constant map $c\colon X \to X$. Let $f\colon X \to Y$ be any map. Then $f = f \circ id_X \simeq f \circ c$, and $f \circ c$ is constant.

4. If X is contractible, then any map from any space Y to X is homotopic to a constant map.

    Since X is contractible, $id_X \simeq c$ for some constant map $c\colon X \to X$. Let $f\colon Y \to X$ be any map. Then $f = id_X \circ f \simeq c \circ f$, and $c \circ f$ is constant.

5. If X is contractible, then two maps $f, g\colon X \to Y$ are homotopic if and only if their images lie in the same path component of Y.

    Assume first that f and g are homotopic, by a homotopy $F\colon X \times I \to Y$. Since X is path-connected, the images $f(X)$ and $f(Y)$ are path-connected, so each is contained in some path component of Y. Choose any $x_0 \in X$. Then $\alpha\colon I \to X$ defined by $\alpha(t) = F(x_0, t)$ is a path from $f(x_0)$ to $g(x_0)$, so $f(X)$ and $g(X)$ must lie in the same path component of Y.

    Conversely, assume that the images lie in the same path component of Y. Then for some $y_0, y_1 \in Y$, we have $f \simeq c_{y_0}$ and $g \simeq c_{y_1}$, where c_y means the constant map that sends X to y. By the previous part, the image y_0 of c_{y_0} lies in the same path component of Y as does $f(X)$, and and y_1 lies in the same path component of Y as does $g(X)$. Therefore y_0 and y_1 lie in the same path component, so there exists a path $\alpha\colon I \to Y$ from y_0 to y_1. The homotopy defined by $F(x,t) = \alpha(t)$ is a homotopy from c_{y_0} to c_{y_1}. Therefore we have $f \simeq c_{y_0} \simeq c_{y_1} \simeq g$.

16. (2/8) Use the Classification Theorem to prove that a (compact, connected) surface is planar if and only if it is a connected sum of disks. Hint: First show that if F has a torus or projective plane summand, then it contains a pair of imbedded circles that intersect at only one point, at which they cross. Use the Jordan Curve Theorem to show that $\mathbb{R}^2$ contains no such pair. Use these two facts, plus the Classification Theorem, to show that the only surfaces that imbed in $\mathbb{R}^2$ are connected sums of disks.

Suppose that F is planar, so $F \neq S^2$ and F can be imbedded in S^2. The torus T contains two loops that cross in one point, for example regard T as $S^1 \times S^1$ and take the circles $S^1 \times \{s_0\}$ and $\{s_0\} \times S^1$ for some $s_0 \in S^1$; these cross only in the point (s_0, s_0). The projective plane P also contains such a pair of loops. For example, regard P as D with the identifications $x \sim -x$ for x in the boundary circle $\partial D = S^1$, and take the loops $([-1,1] \times \{0\})/(-1,0) \sim (1,0)$ and $(\{0\} \times [-1,1])/(0,-1) \sim (0,1)$, which cross only at $(0,0)$. Since the disk removed to form the connected sum may be chosen to miss the two loops that intersect in one point, any surface that is a connected sum $T\#M$ or $P\#M$ will contain such a pair of loops. So, if we prove that S^2 contains no such pair of loops, then no surface with a T or P summand can be planar, so by the classification theorem a planar surface can only have D summands.

The Jordan Curve theorem says, among other things, that any imbedded circle C in S^2 or $\mathbb{R}^2$ has two complementary components, say A and B. Crossing C, one goes either from A to B or from B to A, so any path that starts and ends in the same component must cross C an even number of times. In particular, any path that starts and ends at the same point must cross C an even number of times, so any imbedded circle must cross C an even number (possibly 0) of times. Thus S^2 contains no pair of circles that cross only once, and as already noted this shows that a planar surface has no T or P summand.

Conversely, since D does imbed in S^2 and $kD = D\#\cdots\#D$ is obtained from D by removing open disks from D, kD also imbeds in S^2, any connected sum of disks is planar.