8. (2/1) Suppose that \( h_0, h_1: X \rightarrow Y \) are isotopic. Prove that if \( g: Y \rightarrow Z \) is a homeomorphism, then \( g \circ h_0 \simeq g \circ h_1 \). Prove that if \( k: Z \rightarrow X \) is a homeomorphism, then \( h_0 \circ k \simeq h_1 \circ k \).

Let \( h_t \) be the isotopy. Each \( g \circ h_t \) is a homeomorphism, so \( g \circ h_t \) defines an isotopy from \( g \circ h_0 \) to \( g \circ h_1 \), and similarly \( h_t \circ k \) defines an isotopy from \( h_0 \circ k \) to \( h_1 \circ k \).

If one wishes to worry about the continuity, just note that the isotopy \( g \circ h_t \) is the composition of the map \( X \times I \rightarrow Y \times I \), defined by sending \((x, t)\) to \((h_t(x), t)\) (which is continuous since \( h_t \) is an isotopy) and the map \( g \circ \pi_1: Y \times I \rightarrow Z \).

9. (2/1) An imbedding \( j: I \rightarrow I \) is called order-preserving if \( j(0) < j(1) \), otherwise it is called order-reversing.

1. Prove that if \( j \) is order-preserving, then \( j(x_1) < j(x_2) \) whenever \( x_1 < x_2 \).

Suppose that \( x_1 < x_2 \) but \( j(x_1) > j(x_2) \). If \( j(0) < j(x_2) \), then the Intermediate Value Theorem applied to the interval \([0, x_1]\) produces \( c < x_1 \) with \( j(c) = j(x_2) \). If \( j(x_2) < j(0) < j(x_1) \), then the Intermediate Value Theorem applied to the interval \([x_1, x_2]\) produces \( c > x_1 \) with \( j(0) = j(c) \). If \( j(x_1) < j(0) \), then the Intermediate Value Theorem applied to the interval \([x_2, 1]\) produces \( c > x_2 \) with \( j(x_1) = j(c) \). In any case, we contradict the fact that \( j \) is an imbedding.

2. Prove that there are exactly two isotopy classes of imbeddings of \( I \) into \( I \), by showing that \( j_0, j_1: I \rightarrow I \) are isotopic if and only if they are both order-preserving or both order-reversing.

Suppose first that \( j_0 \) and \( j_1 \) are both order-preserving. Define \( j: I \times I \rightarrow I \) by \( j(x, t) = (1 - t)j_0(t) + tj_1(t) \). Since \( j \) is clearly continuous, and \( j(x, 0) = j(0) \) and \( j(x, 1) = j(1) \), it suffices to show that each \( j_i \) is injective. Then, since \( I \) is compact Hausdorff, we will know automatically that each \( j_i \) is an imbedding, so \( j \) will be an isotopy of imbeddings from \( j_0 \) to \( j_1 \).

By the previous part of the problem, we know that if \( x_1 < x_2 \), then \( j_0(x_1) < j_0(x_2) \) and \( j_1(x_1) < j_1(x_2) \). Suppose that for some \( t \) and some \( x_1 < x_2 \) we have \( j_t(x_1) = j_t(x_2) \). Using the definition of \( j_t \) shows that \((1 - t)(j_0(x_2) - j_0(x_1)) = -t(j_1(x_2) - j_1(x_1)) \). This is a contradiction, since the left-hand side is positive and the right-hand side is negative.

Suppose now that \( j_0 \) and \( j_1 \) are both order-reversing. Let \( \rho: I \rightarrow I \) be the reflection homeomorphism defined by \( \rho(x) = 1 - x \). Then \( j_0 \circ \rho \) and \( j_1 \circ \rho \) are order-preserving, so the previous case shows that \( j_0 \circ \rho \) and \( j_1 \circ \rho \) are isotopic. Problem 8 shows that \( j_0 \circ \rho \circ j_0 \circ \rho \) and \( j_1 \circ \rho \circ \rho \) are isotopic. Since \( \rho \circ \rho = id_I \), this shows that \( j_0 \) and \( j_1 \) are isotopic.

Finally, we must show that if \( j_0 \) is order-preserving and \( j_1 \) is order-reversing, then \( j_0 \) is not isotopic to \( j_1 \). If \( j_t \) were an isotopy, then the Intermediate Value Theorem applied to the function \( j_t(0) - j_t(1) \) produces a \( t_0 \) with \( j_{t_0}(0) = j_{t_0}(1) \), contradicting the fact that \( j_{t_0} \) is an imbedding.
10. (2/1) Prove the Disk Lemma for $n = 1$ and $M = I$. That is, prove that if $j_1, j_2 : I \to I$ are imbeddings with image in the interior of $I$, then $j_1$ is ambiently isotopic to either $j_2$ or $j_2 \circ \rho$. Hint: this follows quickly from the fact that any two homeomorphisms of $I$ are isotopic. Compose $j_1$ and/or $j_2$ by $\rho$ to assume that both are order preserving. Extend the homeomorphism $j_2 \circ j_1^{-1} : j_1(I) \to j_2(I)$ to an order-preserving homeomorphism $h : I \to I$, by using linear maps on $I - j_1(I)$. Now make use of the fact that $id_I$ and $h$ are isotopic.

Assume first that both $j_1$ and $j_2$ are order-preserving. We have a homeomorphism $j_2 \circ j_1^{-1} : j_1(I) \to j_2(I)$. We extend this to a homeomorphism $h : I \to I$ using order-preserving linear homeomorphisms from $[0, j_1(0)]$ to $[0, j_2(0)]$ and from $[j_1(1), 1]$ to $[j_2(1), 1]$. Explicitly, for $x \in [0, j_1(0)]$ put $h(x) = xj_1(0)/j_2(0)$ (using the fact that both $j_1(0), j_2(0) > 0$, and for $x \in [j_1(1), 1]$ put $h(x) = 1 - j_2(1) + (x - j_1(1))/(-j_1(1))$ (using the fact that $j_1(1), j_2(1) < 1$). Since $h$ is an order-preserving homeomorphism, there is an isotopy $H$ from $id_I$ to $h$. Then, $H_0 = id_I$ and $H_1 \circ j_1 = h \circ j_1 = j_2 \circ j_1^{-1} \circ j_1 = j_2$, showing that $j_1$ and $j_2$ are ambiently isotopic.

11. (2/1) A compact (connected) surface $F$ is called planar if $F \neq S^2$ and $F$ can be imbedded into $S^2$. Show that if $F_1$ and $F_2$ are planar, then the connected sum $F_1 \# F_2$ is planar. Hint: Let $D_1 \subset F_1$ and $D_2 \subset F_2$ be admissible disks. Use the Disk Lemma to show that there is an imbedding of $F_1$ in $S^2$ that carries $D_1$ to the upper hemisphere, and there is an imbedding of $F_2$ in $S^2$ that carries $D_2$ to the lower hemisphere.

Fix imbeddings $f_1 : F_1 \to S^2$ and $f_2 : F_2 \to S^2$. Let $i_+: D^2 \to S^2$ be the imbedding $i_+(x,y) = (x, y, \sqrt{1 - x^2 - y^2})$, and let $i_- : D^2 \to S^2$ be the imbedding $i_-(x,y) = (x, y, -\sqrt{1 - x^2 - y^2})$. Finally, let $i_1 : D^2 \to F_1$ and $i_2 : D^2 \to F_2$ be any (admissible) imbeddings. By the Disk Lemma, there are ambient isotopies $J_1$ and $J_2$ of $S^2$ with $(J_1)_1 \circ f_1 \circ i_1 = i_+ \circ \rho^1$ and $(J_2)_1 \circ f_2 \circ i_2 = i_- \circ \rho^2$, where each of $\epsilon_1$ and $\epsilon_2$ is either 0 or 1. Denote $(J_i)_1 \circ f_i$ by $k_i$. Notice that $i_+(D^2) = i_+ \circ \rho^1(D^2) = k_1(i_1(D^2))$, so $i_+(D^2)$ is in the interior of $k_1(F_1)$, and similarly $i_-(D^2)$ is in the interior of $k_2(F_2)$.

We may define $F_1 \# F_2$ using any (admissible) imbeddings of $D^2$ into $F_1$ and $F_2$, so assume it is defined using the imbeddings $j_1 = k_1^{-1} \circ i_+ : D^2 \to F_1$ and $j_2 = k_2^{-1} \circ i_- : D^2 \to F_2$. That is, let $F_1 \# F_2$ be the identification space obtained from the union of $F_1 - j_1(\text{int}(D^2))$ and $F_1 - j_1(\text{int}(D^2))$ by identifying $j_1(p)$ with $j_2(p)$ for each $p \in \partial D^2$. Now, $F_1 \# F_2 \neq S^2$, for if so then when we cut $S^2$ along $\partial D^2$, we obtain two $D^2$s, and then each of $F_1$ and $F_2$ would have to be $S^2$. It remains to show that $F_1 \# F_2$ can be imbedded in $S^2$. Define $f : F_1 \# F_2 \to S^2$ by $f(z) = k_1(z)$ for $z \in F_1 - j_1(\text{int}(D^2))$ and $f(z) = k_2(z)$ for $z \in F_2 - j_2(\text{int}(D^2))$. For $p \in \partial D^2$, we have $f(j_1(p)) = k_1 \circ j_1(p) = k_1 \circ k_1^{-1} \circ i_+(p) = i_+(p)$ and $f(j_2(p)) = k_2 \circ j_2(p) = k_2 \circ k_2^{-1} \circ i_-(x) = i_-(p)$. Since $i_+(p) = i_-(p)$ for $p \in \partial D^2$, this shows $f$ is well-defined. It is continuous by gluing on closed sets, and is injective by construction. Since it is a continuous injection from a compact space into a Hausdorff space, it is an imbedding. Therefore $F_1 \# F_2$ is planar.