Math 5863 homework solutions

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

4. Prove that the relation $\simeq$ of being homotopic is an equivalence relation on the set of continuous maps from X to Y.

   For $f\colon X \to Y$, putting $F(x,t) = f(x)$ defines a homotopy from f to f. If $F\colon f \simeq g$, define $\overline{F}\colon X \times I \to Y$ by $\overline{F}(x,t) = F(x, 1-t)$, then $\overline{F}(x,0) = F(x,1) = g(x)$ and similarly $\overline{F}(x,1) = f(x)$, so $\overline{F}\colon g \simeq f$. Suppose that $F\colon f \simeq g$ and $G\colon g \simeq h$. Define $F*G\colon X \times I \to Y$ by $F*G(x,t) = F(x,2t)$ if $0 \leq t \leq 1/2$ and $F*G(x,t) = F(x,2t-1)$ if $1/2 \leq t \leq 1$. By patching of continuous functions on the closed sets $X \times [0,1/2]$ and $X \times [1/2,1]$, $F*G$ is continuous, and $F*G\colon f \simeq h$.

5. Let X be a one-point space, $X = \{*\}$. Prove that the homotopy classes of continuous maps from X to Y correspond bijectively to the path components of Y.

   For $f\colon X \to Y$ denote the homotopy class of f by $[f]$, and for $y \in Y$ denote the path component of y by $\langle y \rangle$. Define Φ from the set of homotopy classes of maps from X to Y to the set of path components of Y by $\Phi([f]) = \langle f(*) \rangle$. This is well-defined, since if $F\colon f \simeq g$, then sending t to $F(*,t)$ is a path from $f(*)$ to $g(*)$, so $\langle f(*) \rangle = \langle g(*) \rangle$. For each $y \in Y$, the function defined by $f(*) = y$ is continuous, since X has the discrete topology, and $\Phi([f]) = \langle y \rangle$, so Φ is surjective. If $\Phi([f]) = \Phi([g])$, then there is a path α from $f(*)$ to $g(*)$, and putting $F(x,t) = \alpha(t)$ defines a homotopy from f to g, so Φ is injective.

6. Suppose that $f_0, f_1\colon X \to Y$ are homotopic. Prove that if $g\colon Y \to Z$ is a continuous map, then $g \circ f_0 \simeq g \circ f_1$. Prove that if $k\colon Z \to X$ is a continuous map, then $f_0 \circ k \simeq f_1 \circ k$.

   Let $F\colon f_0 \simeq f_1$ be a homotopy. Then $g \circ F\colon X \times I \to Z$ is a homotopy from $g \circ f_0$ to $g \circ f_1$, and the map $G\colon Z \times I \to Y$ defined by $G(x,t) = F(k(x),t)$ is a homotopy from $f_0 \circ k$ to $f_1 \circ k$ (G is continuous since it is the composition of the map $k \times id_I\colon Z \times I \to X \times I$ and the original homotopy F).

7. Recall that the *cone* on X, $C(X)$, is the quotient space obtained by identifying the subspace $X \times \{1\}$ of $X \times I$ to a point. We identify X with the subspace $X \times \{0\}$ of $C(X)$, by letting x correspond to the point $[(x,0)]$. Let $f\colon X \to Y$ be a continuous map. Prove that f is homotopic to a constant map if and only if there exists a continuous map $g\colon C(X) \to Y$ for which $g|_X = f$.

Let $H\colon f \simeq c$ be a homotopy from f to a constant map $c\colon X \to Y$, and let $q\colon X \times I \to C(X)$ be the quotient map. Since $H(x,1) = c(x) = H(y,1)$ for every $x, y \in X$, H is constant on the point preimages of q. By the universal property of the quotient topology, H induces a continuous map $g\colon C(X) \to Y$, and $g([x,0]) = H(x,0) = f(x)$. Conversely, if $g\colon C(X) \to Y$ exists define $H\colon X \times I \to Y$ by $H = g \circ q$, then $H(x,0) = g([x,0]) = f(x)$ and $H(x,1) = g([x,1]) = c(x)$ for all x.