Math 5863 homework solutions

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

1. (1/18) The Klein bottle \( K \) can be constructed from two annuli \( A_1 \) and \( A_2 \) by identifying their boundaries in a certain way. For each of the three descriptions of \( K \) discussed in class (two Möbius bands with boundaries identified, the square with certain identifications on its boundary, and \( S^1 \times I \) with the two ends identified), make a drawing showing where \( A_1 \) and \( A_2 \) appear in \( K \).

2. (1/18) Two surfaces \( F_1 \) and \( F_2 \) can be constructed as follows. Start with \( S^1 \times I \), and remove the interior of a small disk \( D \) from the interior of \( S^1 \times I \). For \( F_1 \), identify each \((\theta, 0)\) with \((\theta, 1)\) and identify each point of \( \partial D \) with its antipodal point (that is, if \( \partial D \) is regarded as \( S^1 \), then \( v \) is identified with \(-v\)). For \( F_2 \), identify each \((\theta, 0)\) with \((\theta, 1)\) and identify each point of \( \partial D \) with its antipodal point.

   1. Make drawings illustrating each of \( F_1 \) and \( F_2 \). Notice that both are closed surfaces.
   2. Find three disjoint Möbius bands imbedded in \( F_1 \).
   3. Find three disjoint Möbius bands imbedded in \( F_2 \).

Actually, \( F_1 \) and \( F_2 \) are homeomorphic, although this may not be very easy to see.
3. (1/18) Let $M$ and $N$ be $n$-dimensional manifolds, and let $U$ be an open subset of $M$. Suppose that $f: U \to N$ is a continuous injection. Prove that $f$ takes open sets in $U$ to open sets in $N$.

Solution 1: Let $V$ be an open subset of $U$. It suffices to show that for any point $f(x)$, there exists an open set $J$ in $N$ with $f(x) \in J \subseteq f(V)$. Choose an open neighborhood $W_{f(x)}$ of $f(x)$ with $W_{f(x)}$ homeomorphic to $\mathbb{R}^n$. Choose an open neighborhood $V_x$ of $x$ with $V_x \approx \mathbb{R}^n$. Now $f^{-1}(W_{f(x)})$ is an open neighborhood of $x$, and $f$ carries the open subset $f^{-1}(W_{f(x)}) \cap V \cap V_x$ of $x$ by a continuous injection into $W_{f(x)}$. By Invariance of Domain, $J = f(f^{-1}(W_{f(x)}) \cap U \cap V_x)$ is open in $W_{f(x)}$, and hence in $N$, and $f(x) \in J \subseteq W_{f(x)}$.

Solution 2: Let $V$ be open in $U$. It is an open subset of a manifold, hence is a manifold, so we may select a collection $\{V_\alpha\}$ of charts whose union is $V$. Let $\{W_\beta\}$ be a collection of charts whose union is $N$. We have

$$f(V) = \cup_\alpha f(V_\alpha) = (\cup_\alpha f(V_\alpha)) \cap (\cup_\beta W_\beta)$$

$$= \cup_\alpha (f(V_\alpha) \cap (\cup_\beta W_\beta)) = \cup_{\alpha, \beta} f(V_\alpha \cap f^{-1}(W_\beta))$$

Each $V_\alpha \cap f^{-1}(W_\beta)$ is open in $V_\alpha$, so by Invariance of Domain its image $f(V_\alpha \cap f^{-1}(W_\beta))$ is an open subset of $W_\beta$, and hence is open in $N$. So $f(V)$ is a union of open subsets of $N$. 