Math 5863 homework

45. Let $p_E\colon (E,e_0)\to(B,b_0)$ be a covering map between path-connected, locally connected spaces. Let $f\colon S^1\to B$ be continuous. Prove that there exists a map $F\colon \mathbb{R}\to E$ such that $p_E\circ F = f\circ p$, where $p\colon \mathbb{R}\to S^1$ is the usual covering map.

    Since $\pi_1(\mathbb{R})=\{1\}$, $(f\circ p)_\#(\pi_1(\mathbb{R}))\subseteq (p_E)_\#(\pi_1(E,e_0))$. Therefore there exists a lift $F\colon \mathbb{R}\to E$ of $f\circ p$, that is, $p_E\circ F = f\circ p$.

46. (4/21) Let X be path-connected. Prove that if $\pi_1(X)$ is finite, then any map from X to S^1 is homotopic to a constant map.

    Let $f\colon X\to S^1$, so $f_\#\colon \pi_1(X,x_0)\to\pi_1(S^1,f(x_0))\cong\mathbb{Z}$. Any homomorphism must take an element of finite order to an element of finite order (since if $g^n=1$, then $(\phi(g))^n=\phi(g^n)=\phi(1)=1$). The only element of finite order in $\mathbb{Z}$ is the identity element 0, so $f_\#$ must take every element to 0. Therefore $f_\#(\pi_1(X,x_0))\subseteq p_\#(\pi_1(\mathbb{R},r_0))$ (where r_0 is any point with $p(r_0)=f(x_0)$). By the Lifting Criterion, there exists a lift $F\colon X\to\mathbb{R}$, and since $\mathbb{R}$ is contractible, F is homotopic to a constant map c. Therefore $f=p\circ F\simeq p\circ c$, which is constant.

47. (4/21) Let $p\colon (E,e_0)\to(B,b_0)$ be a covering map between path-connected, locally connected spaces. Prove that if $p_\#\colon \pi_1(E,e_0)\to\pi_1(B,b_0)$ is surjective, then p is a homeomorphism.

    Assume that $p_\#\colon \pi_1(E,e_0)\to\pi_1(B,b_0)$ is surjective. Then $id_B\colon B\to B$ satisfies $(id_B)_\#(\pi_1(B,b_0))=\pi_1(B,b_0)\subseteq p_\#(\pi_1(E,e_0))$, so by the Lifting Criterion there exists a map $P\colon (B,b_0)\to(E,e_0)$ with $p\circ P = id_B$.

    We will now show that $P\circ p = id_E$. We have $p\circ P\circ p = id_B\circ p = p$, so $P\circ p$ is a lift of p. Also, id_E is a lift of p. Since $P\circ p(e_0)=P(b_0)=e_0=id_E(e_0)$, $P\circ p$ and id_E agree at a point. By the uniqueness of lifts, $P\circ p = id_E$.

    We have shown that P is a two-sided continuous inverse of p, so p is a homeomorphism.

48. (4/21) Let $p\colon E\to B$ be a covering map, with B and E path-connected and locally path-connected. Let $\tau\colon E\to E$ be a map such that $p\circ\tau = p$. Such a τ is called a *covering transformation*.

    1. Observe that τ is a lift of p. Deduce that if if τ_1 and τ_2 are two covering transformations and $\tau_1(e)=\tau_2(e)$ for some $e\in E$, then $\tau_1=\tau_2$; in particular if $\tau(e)=e$ for some point $e\in E$, then $\tau = id_E$.

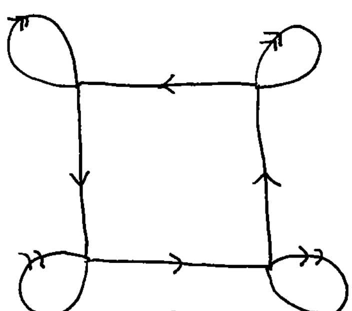
        By definition, a covering transformation $\tau\colon E\to E$ is a map such that $p\circ\tau = p$, which is what it means to say that τ is a lift of $p\colon E\to B$. Since any two lifts of any map that agree at a point are equal, this applies to any two covering transformations. Now id_E is a covering transformation, since $p\circ id_E = p$, so if $\tau(e)=e$, then $\tau(e)=id_E(e)$ and it follows that $\tau = id_E$.

2. Show that the set of covering transformations forms a group under the operation of composition.

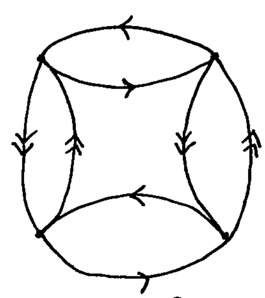
   First we show that τ must be a homeomorphism. Since $p = p \circ \tau$, we have $p_\#(\pi_1(E, \tau(e_0))) = (p \circ \tau)_\#(\pi_1(E, \tau(e_0))$ so by the Lifting Criterion applied to the covering map $p \circ \tau \colon (E, e_0) \to (B, b_0)$ and the map $p \colon (E, \tau(e_0)) \to (B, b_0)$, there exists a lift $\sigma \colon (E, \tau(e_0)) \to (E, e_0)$. That is, $p = p \circ \tau \circ \sigma$. Since $\tau \circ \sigma \colon (E, \tau(e_0)) \to (E, \tau(e_0))$ is a lift of p and fixes the point $\tau(e_0)$, $\tau \circ \sigma = id_E$. Also, $p \circ \sigma \circ \tau = p \circ \tau \circ \sigma \circ \tau$ (since $p = p \circ \tau$), which equals $p \circ \tau$ (since $p = p \circ \tau \circ \sigma$), which equals p, and $\sigma \circ \tau$ fixes e_0 so by the same argument $\sigma \circ \tau = id_E$. Therefore σ is a continuous inverse for τ, so τ is a homeomorphism and its inverse is a covering transformation (which is also automatic since if $p = p \circ \tau$ then $p \circ \tau^{-1} = p \circ \tau \circ \tau^{-1} = p$).

   The composition of covering transformations is a covering transformation, since if $p \circ \tau_1 = p \circ \tau_2 = p$ then $p \circ \tau_1 \circ \tau_2 = p \circ \tau_2 = p$, so composition is an associative operation on the set of covering transformations for which inverses and an identity element id_E exist.

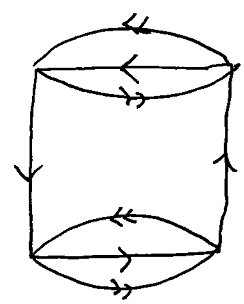
3. Notice that for our examples of covering spaces of $S^1 \vee S^1$, a covering transformation corresponds to an automorphism of the graph which takes single arrows to single arrows, preserving direction, and double arrows to double arrows, preserving direction. Find four 4-fold coverings of $S^1 \vee S^1$ whose groups of covering transformations are respectively C_4, $C_2 \times C_2$, C_2, and $\{1\}$.



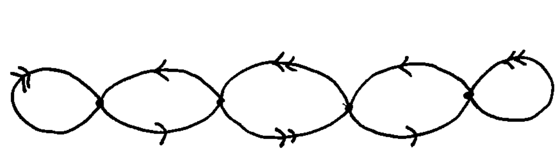
C_4



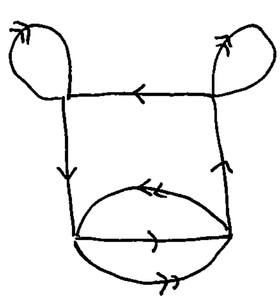
$C_2 \times C_2$



C_2



C_2



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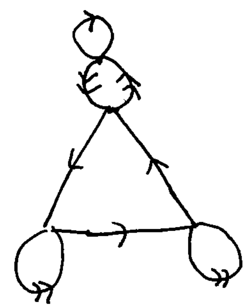

$\{1\}$