Math 5863 homework

40. (4/14) Suppose that $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (X, x_0) \to (Y, y_1)$ are maps, and $H\colon f \simeq g$. Define a path $\gamma\colon I \to Y$ by $\gamma(t) = H(x_0, t)$. Show informally that $f_\# = h_\gamma \circ g_\#$. Deduce that if $y_0 = y_1$, then $f_\#$ and $g_\#$ differ by an inner automorphism.

    The "loops" $\{x_0\}\times[0,s]*\alpha\times\{s\}*\overline{\{x_0\}\times[0,s]}$ in $X\times I$ are loops based at $(x_0,0)$ representing a path homotopy from $\alpha\times\{0\}$ to $\{x_0\}\times[0,1]*\alpha\times\{1\}*\overline{\{x_0\}\times[0,1]}$. Under the homotopy H, the first of these becomes $f\circ\alpha$ and the second becomes $\gamma * g\circ\alpha * \overline{\gamma}$, and under H the intermediate loops become a path homotopy between these two. Therefore $f_\#(\langle\alpha\rangle) = \langle f\circ\alpha\rangle = \langle\gamma * g\circ\alpha*\overline{\gamma}\rangle = h_\gamma\circ g_\#(\langle\alpha\rangle)$, so $f_\# = h_\gamma\circ g_\#$. When $y_0 = y_1$, γ is a loop and $h_\gamma = \mu(\langle\gamma\rangle)$, so $f_\# = \mu(\langle\gamma\rangle)\circ g_\#$.

41. (4/14) Prove that a space X is contractible if and only if X is homotopy equivalent to a space with one point.

    Suppose X is contractible and let $H\colon id_X \simeq c_{x_0}$ for some constant map $c_{x_0}\colon X\to X$ carrying every x to a point x_0. Regard c_{x_0} as the constant map c from X to $\{x_0\}$ followed by the inclusion $i\colon \{x_0\}\to X$. Then $i\circ c = c_{x_0}\simeq id_X$ and $c\circ i = id_{\{x_0\}}$, so c is a homotopy equivalence with homotopy inverse i.

    Conversely, suppose that $X\simeq\{z\}$ for a one-point space. The only map from X to $\{z\}$ is the constant map c, and its homotopy inverse i must include z to some point x_0. So we have $id_X\simeq i\circ c = c_{x_0}$, showing that X is contractible.

42. (4/14) Let $X = S^1\times D^2$, let $p_0 = (1,0)\in S^1$. Let C_1 be the circle consisting of all (θ, p_0), and let C_2 be the circle consisting of all (p_0,θ). Prove that the inclusion $C_1\to X$ is a homotopy equivalence, but the inclusion $C_2\to X$ is not a homotopy equivalence.

    The constant map $c\colon D^2\to\{p_0\}$ is a deformation retraction; a homotopy $H\colon id_{D^2}\simeq i\circ c$ (rel p_0) is defined by $H(x,t) = (1-t)x + tp_0$. Then, $id_{S^1}\times H\colon S^1\times D^2\times I$ is a deformation retraction from $S^1\times D^2$ to C_1, so the inclusion of C_1 is a homotopy equivalence.

    Let $D = \{p_0\}\times D^2\subset S^1\times D^2$. The inclusion $J\colon C_1\to X$ factors as a composition $j\circ i\colon C_2\subset D\subset S^1\times D^2$. Now $D\approx D^2$ so $\pi_1(D) = \{1\}$. So $i_\#$ takes all of $\pi_1(C_1)$, and therefore $J_\# = j_\#\circ i_\#\colon \mathbb{Z}\cong\pi_1(C_2)\to\pi_1(S^1\times D^2)$ is not injective. Since $J_\#$ is not an isomorphism, J cannot be a homotopy equivalence.

43. Let $p\colon E\to B$ be a covering map, with B path-connected.

    1. Prove that if $b_0, b_1\in B$, then $p^{-1}(b_0)$ and $p^{-1}(b_1)$ have the same cardinality. Hint: For a path α in B, define $\Phi_\alpha\colon p^{-1}(\alpha(0))\to p^{-1}(\alpha(1))$ as follows. For $e\in p^{-1}(\alpha(0))$, let $\widetilde{\alpha}$ be the lift of α starting at e, and put $\Phi(e) = \widetilde{\alpha}(1)$. Prove that $\Phi_{\overline{\alpha}}\circ\Phi_\alpha$ is the identity on $p^{-1}(\alpha(0))$. Deduce that Φ_α is a bijection.

For a path α in B, define $\Phi_\alpha\colon p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1))$ as follows. For $e \in p^{-1}(\alpha(0))$, let $\widetilde{\alpha}$ be the lift of α starting at e, and put $\Phi(e) = \widetilde{\alpha}(1)$.

Observe that if $\alpha \simeq_p \beta$, then $\Phi_\alpha = \Phi_\beta$. For a path homotopy between α and β lifts to a path homotopy between $\widetilde{\alpha}$ and $\widetilde{\beta}$, so $\Phi_\alpha(e) = \widetilde{\alpha}(1) = \widetilde{\beta}(1) = \Phi_\beta(e)$.

We further observe that $\Phi_{\alpha*\beta} = \Phi_\beta \circ \Phi_\alpha$. For we have $\widetilde{\alpha * \beta} = \widetilde{\alpha} * \widetilde{\widetilde{\beta}}$, where $\widetilde{\widetilde{\beta}}$ is the lift of β starting at $\widetilde{\alpha}(1)$, and therefore $\Phi_{\alpha*\beta}(e) = \widetilde{\alpha * \beta}(1) = \widetilde{\widetilde{\beta}}(1) = \Phi_\beta(\widetilde{\alpha}(1)) = \Phi_\beta(\Phi_\alpha(e))$. Also, $\Phi_{c_{b_0}}(e) = e$ for all $e \in p^{-1}(b)$, since $\widetilde{c_{b_0}} = c_e$.

Combining the previous observations, we have $\Phi_{\overline{\alpha}} \circ \Phi_\alpha = \Phi_{\alpha * \overline{\alpha}} = \Phi_{c_{b_0}} = id_{p^{-1}(b_0)}$ and similarly $\Phi_\alpha \circ \Phi_{\overline{\alpha}} = id_{p^{-1}(b_1)}$, so Φ_α is a bijection.

2. Prove that if E is compact, then p is finite-to-one.

   Let $b \in B$ and choose an evenly covered neighborhood U_b of b, so that $p^{-1}(U_b) = \cup_{\gamma\in\Gamma} V_\gamma$ with each $p|_{V_\gamma}\colon V_\gamma \to U_b$ a homeomorphism. In particular, each V_γ contains exactly one point of $p^{-1}(b)$. Now, $p^{-1}(b)$ is closed, since $\{b\}$ is a closed subset of B (which is assumed Hausdorff) and p is continuous. So $p^{-1}(b)$ is compact, and therefore there is a finite subcollection $V_{\gamma_1}, \ldots, V_{\gamma_k}$ whose union contains $p^{-1}(b)$. Since each V_{γ_i} contains only one point of $p^{-1}(b)$, $p^{-1}(b)$ must be finite.

44. Let $p\colon \widetilde{G} \to G$ be a covering map, where $\widetilde{G}$ and G are compact 2-manifolds. It is a fact that that $\chi(\widetilde{G}) = k\chi(G)$, where p is k-fold. Here is the idea: Start with a triangulation of G. If necessary, it may be repeatedly subdivided so that each vertex, edge, and face is small enough that it lies in some evenly covered neighborhood. Using the evenly covered property, the preimage of each vertex then consists of exactly k vertices of $\widetilde{G}$, and similarly for edges and faces. So when we use the preimage triangulation to compute $V - E + F$ for $\widetilde{G}$, we obtain exactly k times the corresponding sum for G. Use the fact to:

    1. Show that S^2 is a covering space only of S^2 and $\mathbb{RP}^2$.

       If $S^2 \to B$ is a covering map, then since S^2 is compact, it must be finite-to-one, say k-fold. So $\chi(B) = \chi(S^2)/k > 0$. The only surfaces with positive Euler characteristic are S^2, $\mathbb{RP}^2$, and D^2; S^2 is a covering space of S^2 (by the identity map) and of $\mathbb{RP}^2$, from homework problem #36, but is not a covering space of D^2 since D^2 has nonempty boundary.

    2. Show that the Möbius band is a covering space only of itself.

       Since homeomorphisms carry boundary points of manifolds to boundary points, one may check that a covering map between manifolds restricts to a covering map of their boundaries. If $M \to B$ is a covering map, then $\chi(B) = k\chi(M) = 0$, so B is the annulus, Möbius band, torus, or Klein bottle. The torus and Klein bottle are impossible, since they have no boundary, and the annulus is impossible since it has two boundary circles and M has only one.

    3. Show that the annulus is a covering space only of itself and of the Möbius band.

As in the previous case, the Euler characteristic shows that the annulus can cover only the annulus or the Möbius band. The annulus does cover the Möbius band, for example by the quotient map of $S^1 \times I \to S^1 \times I/\sim$ where $(x, t) \sim (x + \pi, 1 - t)$.

4. Show that the torus is a covering space only of the torus and the Klein bottle.

   As in the previous case, the Euler characteristic shows that the torus can cover only the torus or the Klein bottle. The torus does cover the Klein bottle, for example, regard the torus as $S^1 \times I/\sim$ where $(z, 0) \sim (z, 1)$ and the Klein bottle as $S^1 \times I/\sim$ where $(z, 0) \sim (\overline{z}, 1)$, and let $p\colon T \to K$ be $p(z, t) = (z, 2t)$ for $0 \leq t \leq 1/2$ and $p(z, t) = (z, 2t - 1)$ for $1/2 \leq t \leq 1$.

5. Show that each compact surface is a covering space of only finitely many other surfaces.

   Suppose that $p\colon E \to B$ is a k-fold covering map. Then $\chi(B) = \chi(E)/k$ so $|\chi(B)| \leq |\chi(E)|$. Since there are only finitely many compact connected surfaces of a given Euler characteristic, and there are only finitely many integers less than or equal to $|\chi(E)|$, there are only finitely many possibilties for B.

6. Show that any covering map with $E = T\#D$, $P\#P\#P$, $\mathbb{RP}^2$, or D^2 is a homeomorphism.

   For all of these surfaces $|\chi(E)| = 1$. Since $|\chi(B)| = |\chi(E)|/k$ for a k-fold covering map $E \to B$, and $\chi(B)$ is an integer, the only possibility for k is 1. Since p is then a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism. (Remark: any 1-fold covering map is a homeomorphism, whether E is compact or not.)