Math 5863 homework

44. (4/14) Let $p\colon \widetilde{G} \to G$ be a covering map, where $\widetilde{G}$ and G are compact 2-manifolds. It is a fact that that $\chi(\widetilde{G}) = k\chi(G)$, where p is k-fold. Here is the idea: Start with a triangulation of G. If necessary, it may be repeatedly subdivided so that each vertex, edge, and face is small enough that it lies in some evenly covered neighborhood. Using the evenly covered property, the preimage of each vertex then consists of exactly k vertices of $\widetilde{G}$, and similarly for edges and faces. So when we use the preimage triangulation to compute $V-E+F$ for $\widetilde{G}$, we obtain exactly k times the corresponding sum for G. Use the fact to:

    1. Show that S^2 is a covering space only of S^2 and $\mathbb{RP}^2$.
    2. Show that the Möbius band is a covering space only of itself.
    3. Show that the annulus is a covering space only of itself and of the Möbius band.
    4. Show that the torus is a covering space only of the torus and the Klein bottle.
    5. Show that each compact surface is a covering space of only finitely many other surfaces.
    6. Show that any covering map with $E = T\#D$, $P\#P\#P$, $\mathbb{RP}^2$, or D^2 is a homeomorphism.

45. (4/21) Let $p_E\colon (E, e_0) \to (B, b_0)$ be a covering map between path-connected, locally connected spaces. Let $f\colon S^1 \to B$ be continuous. Prove that there exists a map $F\colon \mathbb{R} \to E$ such that $p_E \circ F = f \circ p$, where $p\colon \mathbb{R} \to S^1$ is the usual covering map.

46. (4/21) Let X be path-connected. Prove that if $\pi_1(X)$ is finite, then any map from X to S^1 is homotopic to a constant map.

47. (4/21) Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map between path-connected, locally connected spaces. Prove that if $p_\#\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is surjective, then p is a homeomorphism.

48. (4/21) Let $p\colon E \to B$ be a covering map, with B and E path-connected and locally path-connected. Let $\tau\colon E \to E$ be a map such that $p \circ \tau = p$. Such a τ is called a *covering transformation*.

    1. Observe that τ is a lift of p. Deduce that if if τ_1 and τ_2 are two covering transformations and $\tau_1(e) = \tau_2(e)$ for some $e \in E$, then $\tau_1 = \tau_2$; in particular if $\tau(e) = e$ for some point $e \in E$, then $\tau = id_E$.
    2. Show that the set of covering transformations forms a group under the operation of composition.
    3. Notice that for our examples of covering spaces of $S^1 \vee S^1$, a covering transformation corresponds to an automorphism of the graph which takes single arrows to single arrows, preserving direction, and double arrows to double arrows, preserving direction. Find four 4-fold coverings of $S^1 \vee S^1$ whose groups of covering transformations are respectively C_4, $C_2 \times C_2$, C_2, and $\{1\}$.