Math 5863 homework solutions

37. Let X and Y be spaces, and give the set of continuous functions $\mathcal{C}(X,Y)$ the C-O topology. Show that if Y is Hausdorff, then $C(X,Y)$ is Hausdorff.

    Let $f, g \in C(X,Y)$ with $f \neq g$. Then for some $x_0 \in X$, $f(x_0) \neq g(x_0)$. Let U and V be disjoint open neighborhoods of $f(x_0)$ and $g(x_0)$ respectively. Then $f \in S(\{x_0\}, U)$ and $g \in S(\{x_0\}, V)$, and $S(\{x_0\}, U) \cap S(\{x_0\}, V) = \emptyset$ since no function can map x_0 both into U and into V.

38. Let X and Y be spaces, and give the set of continuous functions $\mathcal{C}(X,Y)$ the C-O topology. Prove that if A is a subspace of X, then the function $\mathcal{C}(X,Y) \to \mathcal{C}(A,Y)$ determined by sending f to $f|_A$ is continuous.

    Write $R\colon \mathcal{C}(X,Y) \to \mathcal{C}(A,Y)$ for the function determined by restriction. Consider any set $S_A(C,U) \subseteq \mathcal{C}(A,Y)$. Since C is a compact subset of A, it is a compact subset of X as well, so $S_X(C,U) \subseteq \mathcal{C}(X,Y)$ is defined. We have $R^{-1}(S_A(C,U)) = S_X(C,U)$, since a function $f\colon X \to Y$ carries C into U if and only if its restriction $R(f)$ does. So if $\cap_{i=1}^n S_A(C_i, U_i)$ is any basic open set in $\mathcal{C}(A,Y)$, we have $R^{-1}(\cap_{i=1}^n S_A(C_i,U_i)) = \cap_{i=1}^n R^{-1}(S_A(C_i,U_i))$, which is a basic open set in $\mathcal{C}(X,Y)$.

39. Let X, Y, and Z be spaces, with Y locally compact Hausdorff, and give all sets of continuous functions the C-O topology. Define the composition function $C(f,g)\colon \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ by $C(f,g) = g \circ f$. Prove that C is continuous. (Take as known the fact that if K is a compact subset of a locally compact Hausdorff space, and W is an open set containing K, then there exists an open set V with $\overline{V}$ compact and $K \subseteq V \subseteq \overline{V} \subseteq W$.)

    Suppose that $C(f,g) \in S(C,U)$. Now $f(C)$ is a compact set in Y, and $g^{-1}(U)$ is an open subset of Y containing $f(C)$. Since Y is locally compact Hausdorff, there exists an open set $V \subseteq Y$ with $\overline{V}$ compact and $f(C) \subseteq V \subseteq \overline{V} \subseteq g^{-1}(U)$. We have $f \in S(C,V) \subseteq \mathcal{C}(X,Y)$ and $g \in S(\overline{V}, g^{-1}(U)) \subseteq \mathcal{C}(Y,Z)$, and $C(S(C,V) \times S(\overline{V}, g^{-1}(U)) \subseteq S(C,U)$. Now, suppose that $\cap_{i=1}^n S(C_i,U_i)$ is any basic open set in $\mathcal{C}(X,Z)$, and that $C(f,g) \in \cap_{i=1}^n S(C_i,U_i)$. Then with notation as before, $(f,g) \in \cap_{i=1}^n (S(C_i,V_i) \times S(\overline{V_i}, g^{-1}(U)) \subseteq C^{-1}(\cap_{i=1}^n S(C_i,U_i))$. This shows that C is continuous.