Math 5863 homework

1. (1/18) The Klein bottle K can be constructed from two annuli A_1 and A_2 by identifying their boundaries in a certain way. For each of the three descriptions of K discussed in class (two Möbius bands with boundaries identified, the square with certain identifications on its boundary, and $S^1 \times I$ with the two ends identified), make a drawing showing where A_1 and A_2 appear in K.

2. (1/18) Two surfaces F_1 and F_2 can be constructed as follows. Start with $S^1 \times I$, and remove the interior of a small disk D from the interior of $S^1 \times I$. For F_1, identify each $(\theta, 0)$ with $(\theta, 1)$ and identify each point of ∂D with its antipodal point (that is, if ∂D is regarded as S^1, then v is identified with $-v$). For F_2, identify each $(\theta, 0)$ with $(\overline{\theta}, 1)$ and identify each point of ∂D with its antipodal point.

    1. Make drawings illustrating each of F_1 and F_2. Notice that both are closed surfaces.
    2. Find three disjoint Möbius bands imbedded in F_1.
    3. Find three disjoint Möbius bands imbedded in F_2.

    Actually, F_1 and F_2 are homeomorphic, although this may not be very easy to see.

3. (1/18) Let M and N be n-dimensional manifolds, and let U be an open subset of M. Suppose that $f\colon U \to N$ is a continuous injection. Prove that f takes open sets in U to open sets in N.

4. (1/25) Prove that the relation $\simeq$ of being homotopic is an equivalence relation on the set of continuous maps from X to Y.

5. (1/25) Let X be a one-point space, $X = \{*\}$. Prove that the homotopy classes of continuous maps from X to Y correspond bijectively to the path components of Y.

6. (1/25) Suppose that $f_0, f_1\colon X \to Y$ are homotopic. Prove that if $g\colon Y \to Z$ is a continuous map, then $g \circ f_0 \simeq g \circ f_1$. Prove that if $k\colon Z \to X$ is a continuous map, then $f_0 \circ k \simeq f_1 \circ k$.

7. (1/25) Recall that the *cone* on X, $C(X)$, is the quotient space obtained by identifying the subspace $X \times \{1\}$ of $X \times I$ to a point. We identify X with the subspace $X \times \{0\}$ of $C(X)$, by letting x correspond to the point $[(x, 0)]$. Let $f\colon X \to Y$ be a continuous map. Prove that f is homotopic to a constant map if and only if there exists a continuous map $g\colon C(X) \to Y$ for which $g|_X = f$.