Mathematics 2423-001H Name (please print) ______________________

Examination III

April 28, 2005

Instructions: The exam might be on the long side, so avoid spending a lot of time on any individual problem unless you have completed all the other problems that you definitely know how to do. That is, grab easy points first.

**I.** Let $\sin^{-1}(x)$ be the inverse of the function $f(x) = \sin(x)$, $-\pi/2 \le x \le \pi/2$.
(10)

1. Find the domain and range of $\sin^{-1}(x)$.

   Its domain is the range of $f(x)$, that is, $-1 \le x \le 1$. Its range is the domain of $f(x)$, that is, $-\pi/2 \le x \le \pi/2$.

2. Sketch the graph of $\sin^{-1}(x)$.

3. Use right triangles to simplify the expressions $\cos(\sin^{-1}(x))$ and $\cot\left(\sin^{-1}\left(\dfrac{\sqrt{x+2}}{x}\right)\right)$.

   $\sin^{-1}(x)$ is an angle in a right triangle whose opposite leg is $x$ and hypotenuse is 1. By the Pythagorean Theorem, the adjacent leg has length $\sqrt{1-x^2}$, giving $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$.

   $\sin^{-1}\left(\dfrac{\sqrt{x+2}}{x}\right)$ is an angle in a right triangle whose opposite leg is $\sqrt{x+2}$ and hypotenuse is $x$. By the Pythagorean Theorem, the adjacent leg has length $\sqrt{x^2-x-2}$, giving $\cot\left(\sin^{-1}\left(\dfrac{\sqrt{x+2}}{x}\right)\right) = \sqrt{x^2-x-2}/\sqrt{x+2}$.

4. Use the chain rule to calculate the derivative of $\sin^{-1}(x)$, and write the corresponding indefinite integral formula.

   Differentiating the equation $\sin(\sin^{-1}(x)) = x$, we obtain $\cos(\sin^{-1}(x)) \cdot \dfrac{d}{dx}(\sin^{-1}(x)) = 1$, and therefore $\dfrac{d}{dx}(\sin^{-1}(x)) = 1/\cos(\sin^{-1}(x)) = 1/\sqrt{1-x^2}$. The corresponding indefinite integral formula is $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$.

**II.** On one $x$-$y$ coordinate system, sketch the graphs of $\sinh(x)$ and $\cosh(x)$. Explain why $(\cosh(t), \sinh(t))$ is
(10) a point on a hyperbola, and on a second $x$-$y$ coordinate system sketch that hyperbola and a typical point of the form $(\cosh(t), \sinh(t))$, indicating what $t$ equals geometrically.

Since $\cosh^2(t) - \sinh^2(t) = 1$, the point $(\cosh(t), \sinh(t))$ satisfies the equation $x^2 - y^2 = 1$, whose graph is a hyperbola. Since $\cosh(x) > 0$, the point $(\cosh(t), \sinh(t))$ lies on the component of the hyperbola with $x > 0$. The region bounded by the straight line from the origin to $(\cosh(t), \sinh(t))$, the hyperbola, and the $x$-axis has area $|t|/2$.

**III.** Use l'Hôpital's rule to calculate the following limits:
(12)

1. $\displaystyle\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)}$.

   $$\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)} = \lim_{x\to 0} \frac{p\sec^2(px)}{p\sec^2(qx)} = \frac{p\cdot 1}{q \cdot 1} = \frac{p}{q}.$$

2. $\lim\limits_{x\to 0}\sin(x)\ln(x)$.

$$\lim_{x\to 0}\sin(x)\ln(x)=\lim_{x\to 0}\frac{\ln(x)}{\csc(x)}=\lim_{x\to 0}\frac{1/x}{-\csc(x)\cot(x)}=\lim_{x\to 0}-\frac{\sin(x)}{x}\tan(x)=-1\cdot 0=0.$$

3. $\lim\limits_{x\to 0}x^{\sqrt{x}}$.

$$\lim_{x\to 0}x^{\sqrt{x}}=\lim_{x\to 0}e^{\ln(x^{\sqrt{x}})}=\lim_{x\to 0}e^{\sqrt{x}\ln(x)}=\lim_{x\to 0}e^{\frac{\ln(x)}{\frac{1}{\sqrt{x}}}}=\lim_{x\to 0}e^{\frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}}=\lim_{x\to 0}e^{-2\sqrt{x}}=e^0=1.$$

**IV.** (15) Calculate the following integrals.

1. $\displaystyle\int x^2\sin(x)\,dx$

Using integration by parts with $u=x^2$, $du=2x\,dx$, $dv=\sin(x)\,dx$, and $v=-\cos(x)$, we find that $\int x^2\sin(x)\,dx=-x^2\cos(x)+\int 2x\cos(x)\,dx$. Integration by parts with $u=2x$, $du=2\,dx$, $dv=\cos(x)\,dx$, and $v=\sin(x)$ makes this equal to $-x^2\cos(x)+2x\sin(x)-\int 2\sin(x)\,dx=-x^2\cos(x)+2x\sin(x)+2\cos(x)+C$.

2. $\displaystyle\int\sin^3(mx)\,dx$

$\int\sin^3(mx)\,dx=\int(1-\cos^2(mx)\sin(mx)\,dx=\int\sin(mx)\,dx-\int\cos^2(mx)\sin(mx)\,dx=-\frac{1}{m}\cos(mx)+\frac{1}{3m}\cos^3(mx)+C.$

3. $\displaystyle\int\sin^2(x)\cos^2(x)\,dx$

$\int\sin^2(x)\cos^2(x)\,dx=\int\frac{1}{2}(1-\cos(2x))\cdot\frac{1}{2}(1+\cos(2x))\,dx=\int\frac{1}{4}(1-\cos^2(2x))\,dx=\int\frac{1}{4}(1-\frac{1}{2}(1+\cos(4x)))\,dx=\int\frac{1}{8}-\frac{1}{8}\cos(4x)\,dx=\frac{x}{8}-\frac{1}{32}\sin(4x)+C.$

**V.** (10) Calculate the following integral by using the substitution $t=\sqrt{2}\tan(\theta)$. Express the answer in terms of $t$: $\displaystyle\int\frac{t^3}{\sqrt{t^2+2}}\,dt$.

$\int\frac{t^3}{\sqrt{t^2+2}}\,dt=\int\frac{2\sqrt{2}\tan^3(\theta)}{\sqrt{2\tan^2(\theta)+2}}\sqrt{2}\sec^2(\theta)\,d\theta=\int\frac{2\sqrt{2}\sec^2(\theta)\tan^3(\theta)}{\sqrt{2}\sec(\theta)}\sqrt{2}\,d\theta=\int 2\sqrt{2}\sec(\theta)\tan^3(\theta)\,d\theta=2\sqrt{2}\int(\sec^2(\theta)-1)\sec(\theta)\tan(\theta)\,d\theta=2\sqrt{2}\int(u^2-1)\,du$, where $u=\sec(\theta)$, thus the integral equals $\frac{2\sqrt{2}}{3}u^3-\frac{2\sqrt{2}}{u}+C=\frac{2\sqrt{2}}{3}\tan^3(\theta)-2\sqrt{2}\tan(\theta)+C$. Now, $t=\sqrt{2}\tan(\theta)$, so $\tan(\theta)=\frac{t}{\sqrt{2}}$. Using a right triangle with one angle equal to $\theta$, the opposite leg $t$ and adjacent leg $\sqrt{2}$, we find that $\sec(\theta)=\sqrt{t^2+2}/\sqrt{2}$, so in terms of $t$ the integral is $\frac{2\sqrt{2}}{3}\frac{(t^2+2)^{3/2}}{2\sqrt{2}}-2\sqrt{2}\frac{\sqrt{t^2+2}}{\sqrt{2}}+C=\frac{(t^2+2)^{3/2}}{3}-2\sqrt{t^2+2}+C.$

**VI.** (12) For each of the following rational functions, write out the partial fraction decomposition. Do not solve for unknown values of the coefficients.

1. $\dfrac{x^3}{x^4-1}$

$$\frac{x^3}{x^4-1} = \frac{x^3}{(x^2+1)(x^2-1)} = \frac{x^3}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{x+1} + \frac{D}{x-1}$$

2. $\dfrac{1}{x^3+2x^2+x}$

$$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x^2+2x+1)} = \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

3. $\dfrac{x^2}{x^3+1}$

We first observe that $x=-1$ is a root of $x^3+1$, so $x+1$ is a factor. Dividing $x^3+1$ by $x+1$ gives $x^3+1=(x^2-x+1)(x+1)$ (or if one knows the general factorization formula $x^n+1=(x+1)(x^{n-1}-x^{n-2}+x^{n-3}-\cdots+(-1)^{n-1})$ from early in our course, it applies here with $n=3$). Since $x^2-x+1$ has discriminant $b^2-4c=-3$, it is irreducible, so we have
$$\frac{x^2}{x^3+1} = \frac{x^2}{(x^2-x+1)(x+1)} = \frac{Ax+B}{x^2-x+1} + \frac{C}{x+1}$$

**VII.** (6) Evaluate $\displaystyle\int \frac{\sec^2(\theta)\tan^2(\theta)}{\sqrt{9-\tan^2(\theta)}}\,d\theta$ by using one of the following formulas from the table of integrals:

1. $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

2. $\displaystyle\int \frac{u^2}{\sqrt{2au-u^2}}\,du = -\frac{u+3a}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + C$

3. $\displaystyle\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$

4. $\displaystyle\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

Substituting $u=\tan(\theta)$, $du=\sec^2(\theta)\,d\theta$, the integral becomes $\displaystyle\int \frac{u^2}{\sqrt{9-u^2}}\,du$, so the third formula applies with $a=3$. It gives the integral to be $\displaystyle -\frac{\tan(\theta)}{2}\sqrt{9-\tan^2(\theta)} + \frac{9}{2}\sin^{-1}\left(\frac{\tan(\theta)}{3}\right) + C$

**VIII**. Suppose that $f(x)$ is a function whose third derivative $f^{(3)}(x)$ exists and is continuous. Define $E_2(h)$ by
(12) the formula $f(a+h) = f(a) + f'(a)h + \frac{1}{2!}f''(a)h^2 + E_2(h)$.

1. Use integration by parts to calculate that $E_2(h) = \displaystyle\int_0^h \tfrac{1}{2!}(h-t)^2\, f^{(3)}(a+t)\, dt$.

$$\int_0^h \frac{1}{2!}(h-t)^2\, f^{(3)}(a+t)\, dt = \frac{1}{2}(h-t)^2\, f^{(2)}(a+t)\Big|_0^h + \int_0^h (h-t)\, f^{(2)}(a+t)\, dt$$

$$= -\frac{h^2}{2}f^{(2)}(a) + (h-t)\, f'(a+t)\Big|_0^h + \int_0^h f'(a+t)\, dt = -\frac{h^2}{2}f^{(2)}(a) - hf'(a) + f(a+h) - f(a) = E_2(h)$$

2. Let $m$ be the minimum and $M$ the maximum of $f^{(3)}$ on the interval $[a, a+h]$. Show that $\frac{1}{3!}h^3\, m \leq E_2(h) \leq \frac{1}{3!}h^3\, M$.

   Since $m \leq f^{(3)}(a+t) \leq M$ for $0 \leq t \leq h$, we have

$$\int_0^h \frac{1}{2!}(h-t)^2\, m\, dt \leq \int_0^h \frac{1}{2!}(h-t)^2\, f^{(3)}(a+t)\, dt \leq \int_0^h \frac{1}{2!}(h-t)^2\, m\, dt\ .$$

   Since $\displaystyle\int_0^h \tfrac{1}{2!}(h-t)^2\, dt = \tfrac{h^3}{3!}$, the previous inequalities say

$$m\frac{h^3}{3!} \leq E_2(h) \leq M\frac{h^3}{3!}\ .$$

3. Use the Intermediate Value Theorem to show that there exists $c$ in $[a, a+h]$ so that $E_2(h) = \frac{1}{3!}f^{(3)}(c)\, h^3$.

   The previous inequalities say that

$$m \leq \frac{3!}{h^3}\, E_2(h) \leq M$$

   so the Intermediate Value Theorem says there exists a $c$ with $a < c < a+h$ for which $f^{(3)}(c) = \frac{3!}{h^3}\, E_2(h)$. That is, $E_2(h) = \frac{1}{3!}\, f^{(3)}(c)\, h^3$.